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KEY

TO

WELLS' ACADEMIC ALGEBRA.

BY

WEBSTER WELLS, S.B.,

PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

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No. 2104

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PREFACE.

IN this volume will be found the complete solutions of the examples and problems contained in my Academic Algebra. Efforts have been made to secure accuracy in the work, and the author will be greatly obliged for any corrections or suggestions of any kind.

The book is intended *for teachers only*.

WEBSTER WELLS.

BOSTON, 1887.

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No. 2104

CHAPTER I.

Art. 40.—Pages 8, 9.

When $a=2,\ b=3,\ c=1,$ and $d=4$:

1. $a^2+2ab-c+d=4+12-1+4=19.$
2. $3a^3-2a^2b+c^3=24-24+1=1.$
3. $5a^2b+4ab^2-27cd=60+72-108=24.$
4. $2a^2+3bc-\dfrac{5}{cd}=8+9-\dfrac{5}{4}=\dfrac{63}{4}=15\tfrac{3}{4}.$
5. $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{d}=\dfrac{2}{3}+3+\dfrac{1}{4}=\dfrac{47}{12}.$
6. $\dfrac{b^a}{a^d}=\dfrac{3^2}{2^4}=\dfrac{9}{16}.$
7. $\dfrac{cd}{b^2}+\dfrac{ab}{c^2}=\dfrac{4}{9}+6=6\tfrac{4}{9}.$
8. $b^d-a^bb^2=3^4-2^3\times 3^2=81-72=9.$
9. $\dfrac{3d^2}{5ac}-\dfrac{2a}{3b^2}=\dfrac{48}{10}-\dfrac{4}{27}=\dfrac{24}{5}-\dfrac{4}{27}=\dfrac{628}{135}.$
10. $\dfrac{a^2}{b^2}+\dfrac{b^2}{c^2}+\dfrac{c^2}{d^2}=\dfrac{4}{9}+9+\dfrac{1}{16}=\dfrac{1369}{144}.$

When $a=4,\ b=2,\ c=3,$ and $d=1$:

11. $a^2(a+b)-2abc=16\times(4+2)-48=96-48=48.$
12. $7a^2+(a-b)(a-c)=112+(4-2)\times(4-3)=112+2=114.$
13. $15a-7(b+c)-d=60-7\times(2+3)-1=60-35-1=24.$
14. $c(a^{b+d}+a^{b-d})=3\times(4^{2+1}+4^{2-1})=3\times(64+4)=204.$

15. $25a^2-7(b^2+c^2)+d^2=400-7\times(4+9)+1=400-91+1=310.$

16. $\dfrac{4}{3a-3c}+\dfrac{8}{3}=\dfrac{4}{12-9}+\dfrac{8}{3}=\dfrac{4}{3}+\dfrac{8}{3}=4.$

17. $\dfrac{25a-30c-d}{b+c}=\dfrac{100-90-1}{2+3}=\dfrac{9}{5}.$

18. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{c^2-b^2}{c^2+b^2}=\dfrac{16+4}{16-4}-\dfrac{9-4}{9+4}=\dfrac{20}{12}-\dfrac{5}{13}=\dfrac{5}{3}-\dfrac{5}{13}=\dfrac{50}{39}.$

When $a=\dfrac{1}{2},\ b=\dfrac{1}{3},\ c=\dfrac{1}{5}$, and $x=2$:

19. $(2a+3b+5c)(8a+3b-5c)(2a-3b+15c)$
$=(1+1+1)\times(4+1-1)\times(1-1+3)=3\times4\times3=36.$

20. $x^3+\left(\dfrac{1}{a}+\dfrac{1}{b}\right)x^2+\left(\dfrac{1}{b}-\dfrac{1}{a}\right)x+\dfrac{2}{b^2}$
$=8+(2+3)\times4+(3-2)\times2+2\times9=8+20+2+18=48.$

21. $x^4-(2a+3b)x^3+(3a-2b)x^2-cx+bc$
$=16-(1+1)\times8+\left(\dfrac{3}{2}-\dfrac{2}{3}\right)\times4-\dfrac{1}{5}\times2+\dfrac{1}{3}\times\dfrac{1}{5}$
$=16-16+\dfrac{10}{3}-\dfrac{2}{5}+\dfrac{1}{15}=\dfrac{45}{15}=3.$

22. $\dfrac{a^2-\dfrac{b}{2}}{8bc-a}-\dfrac{x}{a+b+c}=\dfrac{\dfrac{1}{4}-\dfrac{1}{6}}{\dfrac{8}{15}-\dfrac{1}{2}}-\dfrac{2}{\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{5}}=\dfrac{\dfrac{1}{12}}{\dfrac{1}{30}}-\dfrac{2}{\dfrac{31}{30}}=\dfrac{30}{12}-\dfrac{60}{31}$
$=\dfrac{5}{2}-\dfrac{60}{31}=\dfrac{35}{62}.$

Art. 41. — Pages 9, 10.

1. $5a+2b.$
2. $2x-y^2.$
3. $abc^2d^3.$
4. $3a^3-2a^2b+c^3.$
5. $(x+y)a.$
6. $(x+y)(a-b).$
7. $\dfrac{a^2}{bc}.$
8. $\dfrac{a^2}{b-c}.$
9. $\dfrac{x}{3}+2=3y-11.$
10. $m(a+b)<\dfrac{1}{x^3}.$

Art. 43.—Page 12.

4. Let $x =$ the less number.
Then $5x =$ the greater number.
By the conditions, $x + 5x = 42.$
Or, $6x = 42.$
Dividing by 6, $x = 7$, the less number.
Whence, $5x = 35$, the greater number.

5. Let $x =$ the no. of years in A's age.
Then $x + 6 =$ the no. of years in B's age.
By the conditions, $x + x + 6 = 68.$
Or, $2x + 6 = 68.$
Whence, $2x = 62.$
Dividing by 2, $x = 31$, the no. of years in A's age.
Whence, $x + 6 = 37$, the no. of years in B's age.

6. Let $x =$ the no. of dollars A receives.
Then $x + 128 =$ the no. of dollars B receives.
By the conditions, $x + x + 128 = 1200.$
Or, $2x + 128 = 1200.$
Whence, $2x = 1072.$
Dividing by 2, $x = 536$, the no. of dollars A receives.
Whence, $x + 128 = 664$, the no. of dollars B receives.

7. Let $x =$ the no. of cents spent.
Then $3x =$ the no. of cents left.
By the conditions, $x + 3x = 372.$
Or, $4x = 372.$
Dividing by 4, $x = 93$, the no. of cents spent.

8. Let $x =$ the no. of dollars A receives.
Then $3x =$ the no. of dollars B receives,
and $9x =$ the no. of dollars C receives.
By the conditions, $x + 3x + 9x = 260.$
Or, $13x = 260.$
Dividing by 13, $x = 20$, the no. of dollars A receives.
Whence, $3x = 60$, the no. of dollars B receives,
and $9x = 180$, the no. of dollars C receives.

9. Let $x =$ the smaller part.
Then $x + 21 =$ the greater part.
By the conditions, $x + x + 21 = 125.$
Or, $2x + 21 = 125.$
Whence, $2x = 104.$
Dividing by 2, $x = 52$, the smaller part.
Whence, $x + 21 = 73$, the greater part.

10. Let $x =$ the first number.

Then, $3x =$ the second number,

and $3x + 7 =$ the third number.

By the conditions,

$$x + 3x + 3x + 7 = 98.$$

Or, $7x + 7 = 98.$

Whence, $7x = 91.$

Dividing by 7, $x = 13$, the first number.

Whence, $3x = 39$, the second number,

and $3x + 7 = 46$, the third number.

11. Let $x =$ the no. of dollars A has.

Then, $2x =$ the no. of dollars C has,

and $2x - 13 =$ the no. of dollars B has.

By the conditions,

$$x + 2x + 2x - 13 = 127.$$

Or, $5x - 13 = 127.$

Whence, $5x = 140.$

Dividing by 5, $x = 28$, the no. of dollars A has.

Whence, $2x = 56$, the no. of dollars C has,

and $2x - 13 = 43$, the no. of dollars B has.

12. Let $x =$ the no. of dols. in the value of the harness.

Then, $11x =$ the no. of dols. in the value of the horse,

and $11x - 175 =$ the no. of dols. in the value of the carriage.

By the conditions,

$$x + 11x + 11x - 175 = 400.$$

Or, $23x - 175 = 400.$

Whence, $23x = 575.$

Dividing by 23, $x = 25$, the no. of dols. in the value of the harness.

Whence, $11x = 275$, the no. of dols. in the value of the horse,

and $11x - 175 = 100$, the no. of dols. in the value of the carriage.

13. Let $x =$ the first number.

Then, $3x =$ the second number,

and $x + 33 =$ the third number.

By the conditions,

$$x + 3x + x + 33 = 108.$$

Or, $5x + 33 = 108.$

Whence, $5x = 75.$

Dividing by 5, $x = 15$, the first number.
Whence, $3x = 45$, the second number,
and $x + 33 = 48$, the third number.

14. Let $x =$ the first part.
Then, $2x =$ the second part,
and $3x =$ the third part.
By the conditions,
$$x + 2x + 3x = 210.$$
Or, $6x = 210.$
Dividing by 6, $x = 35$, the first part.
Whence, $2x = 70$, the second part,
and $3x = 105$, the third part.

15. Let $x =$ the no. of dollars paid for the hog.
Then, $x + 6 =$ the no. of dollars paid for the sheep,
and $x + 6 + 27$,
or $x + 33 =$ the no. of dollars paid for the cow.
By the conditions,
$$x + x + 6 + x + 33 = 75.$$
Or, $3x + 39 = 75.$
Whence, $3x = 36.$
Dividing by 3, $x = 12$, the no. of dollars paid for the hog.
Whence, $x + 6 = 18$, the no. of dollars paid for the sheep,
and $x + 33 = 45$, the no. of dollars paid for the cow.

Art. 46.—Page 14.

1. At noon, $-1°$; at 6 P.M., $-4°$.
2. A.D. 52.
3. At noon, $-6°$; at 6 P.M., $0°$.
4. 87°.
5. $-\$1000$.
6. 14°.
7. 115°.
8. At 8 A.M., $-1°$; at 9 A.M., $+2°$; at noon, $+11°$.

CHAPTER II.

Art. 54.—Pages 17, 18.

1. **Result,** $11-5=6$.
2. **Result,** $-13+3=-10$.
3. **Result,** $12-1=11$.
4. **Result,** $-4-7=-11$.
5. **Result,** $-2a+7a=5a$.
6. **Result,** $b-3b=-2b$.
7. **Result,** $-11m-8m=-19m$.
8. **Result,** $bc+16bc=17bc$.
9. **Result,** $-2ax+7ax=5ax$.
10. **Result,** $-3a^2b^2-a^2b^2=-4a^2b^2$.
11. **Result,** $12mn^2-19mn^2=-7mn^2$
12. **Result,** $-13abc+22abc=9abc$.

14. **Result,** $7a-a-3a=7a-4a=3a$.
15. **Result,** $-6m+m-11m+5m=6m-17m=-11m$.
16. **Result,** $13ab-7ab-8ab-6ab=13ab-21ab=-8ab$.
17. **Result,** $7n^2-n^2-3n^2+11n^2-10n^2=18n^2-14n^2=4n^2$.
18. **Result,** $13ax^3-ax^3-20ax^3+6ax^3-5ax^3$
 $=19ax^3-26ax^3=-7ax^3$.

20. **Result,** $5ax-11b-ax+6b=4ax-5b$.
21. **Result,** $2a+5b-3c-8b+9c=2a-3b+6c$.
22. **Result,** $5m-2n^2+n-2m-b^2+3n^2=3m+n^2+n-b^2$.
23. **Result,** $3x-y-x+6-8y-2x+4y-5=1-5y$.

Art. 57.—Page 19.

1.
$$\begin{array}{r}2a-7x\\ -a+4x\\ a+\ \ x\\ \hline 2a-2x\end{array}$$

2.
$$\begin{array}{r}-3ab+2cd\\ -7ab+8cd\\ 4ab-6cd\\ \hline -6ab+4cd\end{array}$$

3.
$$\begin{array}{r}-11a-\ \ 5mp^2\\ 8a+11mp^2\\ -\ \ 9a-\ \ 7mp^2\\ \hline -12a-\ \ \ \ mp^2\end{array}$$

4.
$$\begin{array}{rrrr}2a & -3b & +5c & \\ & b & -5c & +2d\\ \hline 2a & -2b & & +2d\end{array}$$

5.
$$\begin{array}{r}9mn^2+\ \ x^2y\\ -\ \ mn^2+3x^2y\\ -6mn^2-7x^2y\\ \hline 2mn^2-3x^2y\end{array}$$

6.
$$\begin{array}{rrr}a^2 & -2ab & +\ b^2\\ a^2 & +2ab & +\ b^2\\ 2a^2 & & -2b^2\\ \hline 4a^2 & & \end{array}$$

7.

$$\begin{array}{rrr} 3a^2 & +2ab & +4b^2 \\ 5a^2 & -8ab & +b^2 \\ -6a^2 & +5ab & -5b^2 \\ \hline 2a^2 & -ab & \end{array}$$

8.

$$\begin{array}{rrrr} 6x^3 & & -7x & -4 \\ & x^2 & -x & -2 \\ -x^3 & -9x^2 & +8x & \\ \hline 5x^3 & -8x^2 & & -6 \end{array}$$

9.

$$\begin{array}{rrrrr} 4mn & +3ab & -4c & & \\ 2mn & -4ab & & +3x & \\ & & & -4x & +3m^2 \\ \hline 6mn & -ab & -4c & -x & +3m^2 \end{array}$$

10.

$$\begin{array}{rrr} 3x & -2y & -z \\ -5x & +6y & -7z \\ -x & -y & +8z \\ 4x & -9y & \\ \hline x & -6y & \end{array}$$

11.

$$\begin{array}{rrrr} 6x & -3y & +7m & \\ -x & +y & & +2n \\ -4x & +2y & -5m & -9n \\ -2x & & +m & \\ \hline -x & & +3m & -7n \end{array}$$

12.

$$\begin{array}{rrrr} 2x^3 & -5x^2 & -x & +7 \\ -6x^3 & +3x^2 & +8x & -2 \\ 3x^3 & & +x & -4 \\ & 2x^2 & -5x & +1 \\ \hline -x^3 & & +3x & +2 \end{array}$$

13.

$$\begin{array}{rrrr} 2a & -3b & & +4d \\ & 2b & +4c & -3d \\ 4a & +4b & -3c & +2d \\ -3a & & +2c & \\ \hline 3a & +3b & +3c & +3d \end{array}$$

14.

$$\begin{array}{rrrr} 2a^3 & -a^2b & & -2b^3 \\ 8a^3 & & -8ab^2 & -3b^3 \\ & 3a^2b & -ab^2 & +b^3 \\ -5a^3 & -2a^2b & +6ab^2 & \\ \hline 5a^3 & & -3ab^2 & -4b^3 \end{array}$$

15.

$$\begin{array}{rrrr} -10a^3 & +6a^2x & -5ax^2 & +4x^3 \\ 6a^3 & +2a^2x & +4ax^2 & +3x^3 \\ & -15a^2x & +19ax^2 & -17x^3 \\ 5a^3 & +7a^2x & -18ax^2 & +6x^3 \\ \hline a^3 & & & -4x^3 \end{array}$$

CHAPTER III.

Art. 62. — Page 21.

1. Result, $11 + 3 = 14$.
2. Result, $-5 - 16 = -21$.
3. Result, $-3 + 8 = 5$.
4. Result, $-17 + 11 = -6$.
5. Result, $10 - 23 = -13$.
6. Result, $11 + 13 = 24$.
7. Result, $27a - 13a = 14a$.
8. Result, $17x + 11x = 28x$.
9. Result, $-13y - 4y = -17y$.
10. Result, $-10mn + 18mn = 8mn$
11. Result, $5a^2b - 14a^2b = -9a^2b$.
12. Result, $9ab + 2ab = 11ab$.
13. Result, $xy + cd$.
14. Result, $17m^3 - 41m^3 = -24m^3$.
15. Result, $-5x - 3$.
16. Result, $-x^2y^2 - 5x^2y^2 = -6x^2y^2$
17. Result, $-70abc + 52abc = -18abc$.
18. Result, $-7m^2 + 8n^2$.
19. Result, $-33x^2y^2 - 19x^2y^2 = -52x^2y^2$.
20. The sum of $9ab$ and $-2ab$ is $7ab$. Result, $5ab - 7ab = -2ab$.
21. The sum of $-11x^3$ and $8x^3$ is $-3x^3$; the sum of $-10x^3$ and $4x^3$ is $-6x^3$. Result, $-3x^3 + 6x^3 = 3x^3$.

Art. 64. — Pages 22, 23.

3.

$$\begin{array}{r} ab + cd - ax \\ 4ab - 3cd - 4ax \\ \hline -3ab + 4cd + 3ax \end{array}$$

4.

$$\begin{array}{r} 7x + 5y - 3a \\ x - 7y + 5a - 4 \\ \hline 6x + 12y - 8a + 4 \end{array}$$

5.

$$\begin{array}{r} a - b + c \\ a + b - c \\ \hline -2b + 2c \end{array}$$

6.

$$\begin{array}{r} a^2 + 2ab + b^2 \\ a^2 - 2ab + b^2 \\ \hline 4ab \end{array}$$

7.

$$\begin{array}{r} 7abc - 11x + 5y - 48 \\ 11abc + 3x + 7y + 100 \\ \hline -4abc - 14x - 2y - 148 \end{array}$$

8.

$$\begin{array}{r} 5m - 3y^2 + 7a - 6 \\ 3m + y^2 - 5a - 7 \\ \hline 2m - 4y^2 + 12a + 1 \end{array}$$

9.

$$\begin{array}{r} 31x^2-3y^2+\ \ ab \\ 17x^2+5y^2-4ab+7 \\ \hline 14x^2-8y^2+5ab-7 \end{array}$$

10.

$$\begin{array}{r} 6a+3b-5c+1 \\ 6a-3b-5c \\ \hline 6b\qquad+1 \end{array}$$

11.

$$\begin{array}{r} 3m-5n+\ \ r-2s \\ -\ \ m+3n+2r-5s \\ \hline 4m-8n-\ \ r+3s \end{array}$$

12.

$$\begin{array}{r} d-3b+\ \ a-\ \ c \\ -5d-\ \ b+4a+2c \\ \hline 6d-2b-3a-3c \end{array}$$

13.

$$\begin{array}{r} m^2+3n^3 \\ -4m^2-6n^3+71x \\ \hline 5m^2+9n^3-71x \end{array}$$

14.

$$\begin{array}{r} -3b+\ \ 4c-\ \ 5d+2x \\ 3a-\ \ b-\ \ 6c+\ \ 8d \\ \hline -3a-2b+10c-13d+2x \end{array}$$

15.

$$\begin{array}{r} -2a+b+\ \ c \\ a-b+\ \ c \\ \hline -\ \ a\qquad+2c \end{array}$$

$$\begin{array}{r} a-b-\ \ c \\ -\ \ a\qquad+2c \\ \hline 2a-b-3c \end{array}$$

16.

$$\begin{array}{r} x^4+2x^3\qquad\quad-3x+\ \ 4 \\ 3x^3+3x^2+5x-\ \ 7 \\ \hline x^4-\ \ x^3-3x^2-8x+11 \end{array}$$

17.

$$\begin{array}{r} 4a^3\qquad\quad-3ab^2-5b^3 \\ 6a^2b-\ \ ab^2+4b^3 \\ \hline 4a^3-6a^2b-2ab^2-9b^3 \end{array}$$

18.

$$\begin{array}{r} 2a^4-3a^3+\ \ a^2\qquad\quad-\ \ 8 \\ -2a^4\qquad\quad-5a^2+6a-11 \\ \hline 4a^4-3a^3+6a^2-6a+\ \ 3 \end{array}$$

19.

$$\begin{array}{r} x^2-2xy+3y^2 \\ xy-4y^2 \\ \hline x^2-\ \ xy-\ \ y^2 \\ 2x^2\qquad\ -\ \ y^2 \\ \hline -\ \ x^2-\ \ xy \end{array}$$

20.

$$\begin{array}{r} x+2y-3z \\ -4x+3y+\ \ z \\ \hline -3x+5y-2z \\ -5x+5y+\ \ z \\ \hline 2x\qquad-3z \end{array}$$

21.

$$\begin{array}{r} -\ \ 5a^4+7a^3-5a^2+\ \ a+3 \\ -11a^4-2a^3-6a^2+2a+9 \\ \hline 6a^4+9a^3+\ \ a^2-\ \ a-6 \end{array}$$

22.

$$\begin{array}{r} -2x^3+3x^2y+6xy^2-7y^3 \\ x^3+8x^2y-2xy^2-9y^3 \\ \hline -3x^3-5x^2y+8xy^2+2y^3 \end{array}$$

23.

$$\begin{array}{r}
2x^3 \qquad - x + 5 \\
x^2 + 8x - 11 \\ \hline
2x^3 + x^2 + 7x - 6
\end{array}$$

$$\begin{array}{r}
x^3 - 9x^2 - 11x \quad \\
-4x^3 + 3x^2 \qquad - 6 \\ \hline
-3x^3 - 6x^2 - 11x - 6
\end{array}$$

$$\begin{array}{r}
2x^3 + x^2 + 7x - 6 \\
-3x^3 - 6x^2 - 11x - 6 \\ \hline
5x^3 + 7x^2 + 18x \quad
\end{array}$$

24.

$$\begin{array}{r}
a^2 + ab + b^2 \\
a^2 - 4ab + 5b^2 \\ \hline
2a^2 - 3ab + 6b^2
\end{array}$$

$$\begin{array}{r}
4a^2 - 2ab + 7b^2 \\
-a^2 + 3ab - 2b^2 \\ \hline
3a^2 + ab + 5b^2
\end{array}$$

$$\begin{array}{r}
2a^2 - 3ab + 6b^2 \\
3a^2 + ab + 5b^2 \\ \hline
-a^2 - 4ab + b^2
\end{array}$$

25.

$$\begin{array}{r}
3x^2 + xy - 5y^2 \qquad - 7y - 2 \\
-2x^2 - 5xy + 2y^2 + 6x \qquad - 8 \\ \hline
5x^2 + 6xy - 7y^2 - 6x - 7y + 6
\end{array}$$

26.

$$\begin{array}{r}
3x^5 - 8x^4 + 3x^3 - 5x^2 - 2x \quad \\
-3x^4 + 4x^3 + 6x^2 - 6x + 2 \\ \hline
3x^5 - 5x^4 - x^3 - 11x^2 + 4x - 2
\end{array}$$

27.

$$\begin{array}{r}
2x^3 - x^2y - 5xy^2 \quad \\
3x^2y - 5xy^2 - 4y^3 \\ \hline
2x^3 + 2x^2y - 10xy^2 - 4y^3
\end{array}$$

$$\begin{array}{r}
-2x^3 - 7x^2y \qquad - 6y^3 \\
-6xy^2 + 5y^3 \\ \hline
-2x^3 - 7x^2y - 6xy^2 - y^3
\end{array}$$

$$\begin{array}{r}
2x^3 + 2x^2y - 10xy^2 - 4y^3 \\
-2x^3 - 7x^2y - 6xy^2 - y^3 \\ \hline
4x^3 + 9x^2y - 4xy^2 - 3y^3
\end{array}$$

28.

$$\begin{array}{r}
a^4 \qquad\qquad\qquad - 1 \\
2a^3 - 10a^2 - 7a \quad \\ \hline
a^4 + 2a^3 - 10a^2 - 7a - 1
\end{array}$$

$$\begin{array}{r}
-3a^4 \qquad + 2a^2 - 5a \quad \\
-5a^3 - 12a^2 \qquad + 3 \\ \hline
-3a^4 - 5a^3 - 10a^2 - 5a + 3
\end{array}$$

$$\begin{array}{r}
a^4 + 2a^3 - 10a^2 - 7a - 1 \\
-3a^4 - 5a^3 - 10a^2 - 5a + 3 \\ \hline
4a^4 + 7a^3 \qquad\qquad - 2a - 4
\end{array}$$

CHAPTER IV.

Art. 69.—Pages 25, 26.

3. $a-(b-c)+(-d+e)=a-b+c-d+e.$

4. $5x-\{2x-3y\}-[-2x+4y]=5x-2x+3y+2x-4y=5x-y.$

5. $a-b+c-\overline{a+b-c}-\overline{c-b-a}$
$=a-b+c-a-b+c-c+b+a=a-b+c.$

6. $m^2-2n+\{a-n+3m^2\}-\overline{5a+3n-m^2}$
$=m^2-2n+a-n+3m^2-5a-3n+m^2=5m^2-6n-4a.$

7. $a^2-b^2-(a^2-2ab+b^2)-[a^2+2ab+b^2]$
$=a^2-b^2-a^2+2ab-b^2-a^2-2ab-b^2=-a^2-3b^2.$

8. $3a-(2a-\{a+2\})=3a-(2a-a-2)=3a-2a+a+2=2a+2.$

9. $a-(b+\{-c+d\}-e)=a-(b-c+d-e)=a-b+c-d+e.$

10. $a-[(-b+c)-(d-e)]=a-[-b+c-d+e]=a+b-c+d-e.$

11. $3x-[2y+\overline{x-y}]+[3y-\overline{2x+y}]$
$=3x-[2y+x-y]+[3y-2x-y]$
$=3x-2y-x+y+3y-2x-y=y.$

12. $14x-(5x-9)-\{4-3x-(2x-3)\}$
$=14x-5x+9-\{4-3x-2x+3\}$
$=14x-5x+9-4+3x+2x-3=14x+2.$

13. $2m-[n-\{3m-(2n-m)\}]=2m-[n-\{3m-2n+m\}]$
$=2m-[n-3m+2n-m]=2m-n+3m-2n+m=6m-3n.$

14. $3x-(5x+[-4x-\overline{y-x}])-(-x-3y)$
$=3x-(5x+[-4x-y+x])+x+3y$
$=3x-(5x-4x-y+x)+x+3y$
$=3x-5x+4x+y-x+x+3y=2x+4y.$

15. $3c+(2a-[5c-\{3a+\overline{c-4a}\}])$
$=3c+(2a-[5c-\{3a+c-4a\}])$
$=3c+(2a-[5c-3a-c+4a])$
$=3c+(2a-5c+3a+c-4a)$
$=3c+2a-5c+3a+c-4a=a-c.$

16. $5a-(4a-\{-3a-[2a-\overline{a-1}]\})$
$=5a-(4a-\{-3a-[2a-a+1]\})$
$=5a-(4a-\{-3a-2a+a-1\})$
$=5a-(4a+3a+2a-a+1)$
$=5a-4a-3a-2a+a-1=-3a-1.$

17. $8x-[5x-(3x-4)-\{7x+(-9x+2)\}]$
$=8x-[5x-3x+4-\{7x-9x+2\}]$
$=8x-[5x-3x+4-7x+9x-2]$
$=8x-5x+3x-4+7x-9x+2=4x-2.$

18. $2m-[3m-\{m-(2m-\overline{3m+4})\}-(5m-2)]$
$=2m-[3m-\{m-(2m-3m-4)\}-5m+2]$
$=2m-[3m-\{m-2m+3m+4\}-5m+2]$
$=2m-[3m-m+2m-3m-4-5m+2]$
$=2m-3m+m-2m+3m+4+5m-2=6m+2.$

19. $c-[2c-(6a-b)-\{c-(5a+2b)-(a-3b)\}]$
$=c-[2c-6a+b-\{c-5a-2b-a+3b\}]$
$=c-[2c-6a+b-c+5a+2b+a-3b]$
$=c-2c+6a-b+c-5a-2b-a+3b=0.$

20. $3a-\{b-[b-(a+b)-\{-b-(b-\overline{a-b})\}]\}$
$=3a-\{b-[b-a-b-\{-b-(b-a+b)\}]\}$
$=3a-\{b-[b-a-b-\{-b-b+a-b\}]\}$
$=3a-\{b-[b-a-b+b+b-a+b]\}$
$=3a-\{b-b+a+b-b-b+a-b\}$
$=3a-b+b-a-b+b+b-a+b=a+2b.$

Art. 71.—Page 26.

2. $a+b+c+d=a-(-b-c-d).$
3. $3a-2b+5c-4d=3a-(2b-5c+4d).$
4. $m^3+5m^2-6m+3=m^3-(-5m^2+6m-3).$
5. $x^3y-x^2y^2-xy^3+y^4=x^3y-(x^2y^2+xy^3-y^4).$
6. $x^4-3x^3+2x^2-5x-8=x^4-3x^3-(-2x^2+5x+8).$
7. $a^2-b^2-c^2+2ab+2ac=a^2-b^2-(c^2-2ab-2ac).$
8. $a+b+c+d=a-(-b-[c+d]).$
$3a-2b+5c-4d=3a-(2b-[5c-4d]).$
$m^3+5m^2-6m+3=m^3-(-5m^2-[-6m+3]).$
$x^3y-x^2y^2-xy^3+y^4=x^3y-(x^2y^2-[-xy^3+y^4]).$
$x^4-3x^3+2x^2-5x-8=x^4-3x^3-(-2x^2-[-5x-8]).$
$a^2-b^2-c^2+2ab+2ac=a^2-b^2-(c^2-[2ab+2ac]).$

CHAPTER V.

Art. 80. — Pages 29, 30.

5. $13 \times -19 = -247.$
6. $-18 \times 12 = -216.$
7. $-22 \times -51 = 1122.$
8. $15 m^6n^6 \times 3 mn = 45 m^7n^7.$
9. $17abc \times -8 abc = -136a^2b^2c^2.$
10. $-17 a^4c^2 \times 3 a^2c^2 = -51 a^6c^4.$
11. $-11 n^2y \times -5 n^6z = 55 n^8yz.$
12. $-6 a^2bc \times a^3bm = -6 a^5b^2cm.$
13. $-12 a^2x \times -2 x^2y = 24 a^2x^3y.$
14. $-2 a^mb^n \times 5 a^3b^n = -10 a^{m+3}b^{2n}.$
15. $3 a^3x^5y^2 \times 11 ax^4y^5 = 33 a^4x^9y^7.$
16. $3 a^mb^n \times -5 a^nb^r = -15 a^{m+n}b^{n+r}.$

18. $5a \times -6b \times 7c = -210 abc.$
19. $-2a^2 \times -11 a^3 \times -9a = -198 a^6.$
20. $-3 ab^2 \times -2 bc^2 \times 7 cd^2 = 42 ab^3c^3d^2.$
21. $4 x^my^n \times -x^ny^nz^5 \times 15 y^2z^r = -60 x^{m+n}y^{2n+2}z^{r+5}.$
22. $-2a \times -3a^2 \times -4a^3 \times -5a^4 = 120 a^{10}.$
23. $-a^2bc \times 2 b^2cd \times -5 a^3cd \times -3 ab^5d^4 = -30 a^6b^8c^3d^6.$
24. $-7 m^nx^2 \times m^nx^ry^2 \times 2x^3 \times -8 my^{2r} = 112 m^{2n+1}x^{r+5}y^{2r+2}.$
25. $6 xy^2 \times -x^3z \times 3 y^4z^2 \times -2 xz^5 \times -4 yz = -144 x^5y^7z^9.$

Art. 81. — Page 31.

3.
$$\begin{array}{l} 3x - 5 \\ 4x \\ \hline 12x^2 - 20x \end{array}$$

4.
$$\begin{array}{l} a^2b + ab^2 \\ -ab \\ \hline -a^3b^2 - a^2b^3 \end{array}$$

5.
$$\begin{array}{l} 8a^2bc - d \\ 5ad^2 \\ \hline 40a^3bcd^2 - 5ad^3 \end{array}$$

6.
$$\begin{array}{l} x^2 - 2x - 3 \\ -4x \\ \hline -4x^3 + 8x^2 + 12x \end{array}$$

7.
$$\begin{array}{l} 3x^2 + 6x - 7 \\ -2x^3 \\ \hline -6x^5 - 12x^4 + 14x^3 \end{array}$$

8.
$$\begin{array}{l} m^2 + mn + n^2 \\ m^2n^2 \\ \hline m^4n^2 + m^3n^3 + m^2n^4 \end{array}$$

9.

$$\begin{array}{l} 3m^2 - 5mn - n^2 \\ -2m \\ \hline -6m^3 + 10m^2n + 2mn^2 \end{array}$$

10.

$$\begin{array}{l} -\ x^4 - 10x^3 + 5 \\ -2x^3 \\ \hline 2x^7 + 20x^6 - 10x^3 \end{array}$$

11.

$$\begin{array}{l} a^2 + 13ab - 6b^2 \\ 4ab^2 \\ \hline 4a^3b^2 + 52a^2b^3 - 24ab^4 \end{array}$$

12.

$$\begin{array}{l} 5 - 6ac - 8a^3 \\ -\ 6a^2c \\ \hline -30a^2c + 36a^3c^2 + 48a^5c \end{array}$$

13.

$$\begin{array}{l} 5x^3 - 4x^2 - 3x + 2 \\ -\ 6x^5 \\ \hline -30x^8 + 24x^7 + 18x^6 - 12x^5 \end{array}$$

14.

$$\begin{array}{l} a^3 - 3a^2b + 3ab^2 - b^3 \\ a^2b^2 \\ \hline a^5b^2 - 3a^4b^3 + 3a^3b^4 - a^2b^5 \end{array}$$

Art. 82.—Pages 32, 33, 34.

4.

$$\begin{array}{rr} 3x + 2 & \\ 5x - 7 & \\ \hline 15x^2 + 10x & \\ -21x & -14 \\ \hline 15x^2 - 11x & -14 \end{array}$$

5.

$$\begin{array}{rr} 6x - 5 & \\ -\ 2x + 3 & \\ \hline -12x^2 + 10x & \\ +18x & -15 \\ \hline -12x^2 + 28x & -15 \end{array}$$

6.

$$\begin{array}{rr} 3a - 2b & \\ -2a + 4b & \\ \hline -6a^2 + 4ab & \\ 12ab & -8b^2 \\ \hline -6a^2 + 16ab & -8b^2 \end{array}$$

7.

$$\begin{array}{rr} -\ 5xy + 3 & \\ -10xy - 6 & \\ \hline 50x^2y^2 - 30xy & \\ 30xy & -18 \\ \hline 50x^2y^2 \qquad & -18 \end{array}$$

8.

$$\begin{array}{ll} b^2 + ab + a^2 & \\ b - a & \\ \hline b^3 + ab^2 + a^2b & \\ \quad - ab^2 - a^2b & - a^3 \\ \hline b^3 \qquad & - a^3 \end{array}$$

9.

$$\begin{array}{rr} 2a^2b - 3ab^2 & \\ 5a^2b + 6ab^2 & \\ \hline 10a^4b^2 - 15a^3b^3 & \\ 12a^3b^3 & - 18a^2b^4 \\ \hline 10a^4b^2 - 3a^3b^3 & - 18a^2b^4 \end{array}$$

10.

$$\begin{array}{l} x^3 + x^2 + x + 1 \\ ax - a \\ \hline ax^4 + ax^3 + ax^2 + ax \\ \qquad - ax^3 - ax^2 - ax - a \\ \hline ax^4 \qquad\qquad\qquad\quad - a \end{array}$$

11.

$$\begin{array}{l} 3x^2 - 2xy - y^2 \\ 2x - 4y \\ \hline 6x^3 - 4x^2y - 2xy^2 \\ \qquad - 12x^2y + 8xy^2 + 4y^3 \\ \hline 6x^3 - 16x^2y + 6xy^2 + 4y^3 \end{array}$$

12.

$$\begin{array}{l} m^2 - mn - 3n^2 \\ 2m^2 - 6mn \\ \hline 2m^4 - 2m^3n - 6m^2n^2 \\ \qquad - 6m^3n + 6m^2n^2 + 18mn^3 \\ \hline 2m^4 - 8m^3n \qquad\quad + 18mn^3 \end{array}$$

13.

$$\begin{array}{l} x^2 + 2x + 1 \\ x^2 - 2x + 3 \\ \hline x^4 + 2x^3 + x^2 \\ \quad - 2x^3 - 4x^2 - 2x \\ \qquad\qquad 3x^2 + 6x + 3 \\ \hline x^4 \qquad\qquad + 4x + 3 \end{array}$$

14.

$$\begin{array}{l} 5a^2 - 3ab + 4b^2 \\ 6a - 5b \\ \hline 30a^3 - 18a^2b + 24ab^2 \\ \qquad - 25a^2b + 15ab^2 - 20b^3 \\ \hline 30a^3 - 43a^2b + 39ab^2 - 20b^3 \end{array}$$

15.

$$\begin{array}{l} 4x^3 + 6x - 7 \\ 2x^2 - 3 \\ \hline 8x^5 + 12x^3 - 14x^2 \\ \qquad - 12x^3 \qquad - 18x + 21 \\ \hline 8x^5 \qquad - 14x^2 - 18x + 21 \end{array}$$

16.

$$\begin{array}{l} a + b - c \\ a - b + c \\ \hline a^2 + ab - ac \\ \quad - ab \qquad - b^2 + bc \\ \qquad\quad ac \qquad + bc - c^2 \\ \hline a^2 \qquad\quad - b^2 + 2bc - c^2 \end{array}$$

17.

$$\begin{array}{l} 2x^2 - 3x + 5 \\ x^2 + x - 1 \\ \hline 2x^4 - 3x^3 + 5x^2 \\ \qquad 2x^3 - 3x^2 + 5x \\ \qquad\qquad - 2x^2 + 3x - 5 \\ \hline 2x^4 - x^3 \qquad + 8x - 5 \end{array}$$

18.

$$\begin{array}{l} 3x^2 - 7x + 4 \\ 2x^2 + 9x - 5 \\ \hline 6x^4 - 14x^3 + 8x^2 \\ \qquad 27x^3 - 63x^2 + 36x \\ \qquad\qquad - 15x^2 + 35x - 20 \\ \hline 6x^4 + 13x^3 - 70x^2 + 71x - 20 \end{array}$$

19.

$$\begin{array}{l} 2x^3 - 3x^2 - 5x - 1 \\ 3x - 5 \\ \hline 6x^4 - 9x^3 - 15x^2 - 3x \\ \qquad - 10x^3 + 15x^2 + 25x + 5 \\ \hline 6x^4 - 19x^3 \qquad + 22x + 5 \end{array}$$

20.

$$
\begin{array}{llllll}
-m^3 & -2m^2 & +6m & -5 & & \\
 & m^2 & -2m & +10 & & \\
\hline
-m^5 & -2m^4 & +6m^3 & -5m^2 & & \\
 & 2m^4 & +4m^3 & -12m^2 & +10m & \\
 & & -10m^3 & -20m^2 & +60m & -50 \\
\hline
-m^5 & & & -37m^2 & +70m & -50
\end{array}
$$

21.

$$
\begin{array}{llllll}
2x^3 & +5x^2 & -8x & -7 & & \\
-3x^2 & -5x & +4 & & & \\
\hline
-6x^5 & -15x^4 & +24x^3 & +21x^2 & & \\
 & -10x^4 & -25x^3 & +40x^2 & +35x & \\
 & & 8x^3 & +20x^2 & -32x & -28 \\
\hline
-6x^5 & -25x^4 & +7x^3 & +81x^2 & +3x & -28
\end{array}
$$

22.

$$
\begin{array}{llll}
a^3b & -a^2b^2 & -4ab^3 & \\
2a^2b & -ab^2 & & \\
\hline
2a^5b^2 & -2a^4b^3 & -8a^3b^4 & \\
 & -a^4b^3 & +a^3b^4 & +4a^2b^5 \\
\hline
2a^5b^2 & -3a^4b^3 & -7a^3b^4 & +4a^2b^5
\end{array}
$$

23.

$$
\begin{array}{lll}
x^{m+2}y & -3xy^{n-1} & \\
4x^{m+5}y^2 & -4x^4y^n & \\
\hline
4x^{2m+7}y^3 & -12x^{m+6}y^{n+1} & \\
 & -4x^{m+6}y^{n+1} & +12x^5y^{2n-1} \\
\hline
4x^{2m+7}y^3 & -16x^{m+6}y^{n+1} & +12x^5y^{2n-1}
\end{array}
$$

24.

$$
\begin{array}{lllll}
x^2 & -xy & +y^2 & & \\
x^2 & +xy & +y^2 & & \\
\hline
x^4 & -x^3y & +x^2y^2 & & \\
 & x^3y & -x^2y^2 & +xy^3 & \\
 & & x^2y^2 & -xy^3 & +y^4 \\
\hline
x^4 & & +x^2y^2 & & +y^4
\end{array}
$$

25.

$$
\begin{array}{lllll}
4a^2 & +2ab & +b^2 & & \\
4a^2 & -2ab & +b^2 & & \\
\hline
16a^4 & +8a^3b & +4a^2b^2 & & \\
 & -8a^3b & -4a^2b^2 & -2ab^3 & \\
 & & 4a^2b^2 & +2ab^3 & +b^4 \\
\hline
16a^4 & & +4a^2b^2 & & +b^4
\end{array}
$$

26.

$$
\begin{array}{lllllll}
6x^4 & -3x^3 & -x^2 & +6x & -2 & & \\
2x^2 & +x & +2 & & & & \\
\hline
12x^6 & -6x^5 & -2x^4 & +12x^3 & -4x^2 & & \\
 & 6x^5 & -3x^4 & -x^3 & +6x^2 & -2x & \\
 & & 12x^4 & -6x^3 & -2x^2 & +12x & -4 \\
\hline
12x^6 & & +7x^4 & +5x^3 & & +10x & -4
\end{array}
$$

27.

$$
\begin{array}{lllllll}
m^4 & -m^3n & +m^2n^2 & -mn^3 & +n^4 & & \\
m^2 & -2mn & -3n^2 & & & & \\
\hline
m^6 & -m^5n & +m^4n^2 & -m^3n^3 & +m^2n^4 & & \\
 & -2m^5n & +2m^4n^2 & -2m^3n^3 & +2m^2n^4 & -2mn^5 & \\
 & & -3m^4n^2 & +3m^3n^3 & -3m^2n^4 & +3mn^5 & -3n^6 \\
\hline
m^6 & -3m^5n & & & & +mn^5 & -3n^6
\end{array}
$$

28.

$$\begin{array}{l}
27x^3 + 9x^2y + 3xy^2 + y^3 \\
9x^2 - 6xy + y^2 \\
\hline
243x^5 + 81x^4y + 27x^3y^2 + 9x^2y^3 \\
\quad - 162x^4y - 54x^3y^2 - 18x^2y^3 - 6xy^4 \\
\qquad\qquad 27x^3y^2 + 9x^2y^3 + 3xy^4 + y^5 \\
\hline
243x^5 - 81x^4y \qquad\qquad - 3xy^4 + y^5
\end{array}$$

29.

$$\begin{array}{l}
a^3 - 3a^2b + 3ab^2 - b^3 \\
a^2 - 2ab + b^2 \\
\hline
a^5 - 3a^4b + 3a^3b^2 - a^2b^3 \\
\quad - 2a^4b + 6a^3b^2 - 6a^2b^3 + 2ab^4 \\
\qquad\qquad a^3b^2 - 3a^2b^3 + 3ab^4 - b^5 \\
\hline
a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5
\end{array}$$

30.

$$\begin{array}{l}
x^2 - xy - xz + y^2 - yz + z^2 \\
x + y + z \\
\hline
x^3 - x^2y - zx^2 + xy^2 - xyz + xz^2 \\
\quad x^2y \qquad - xy^2 - xyz \qquad + y^3 - y^2z + yz^2 \\
\qquad xz^2 \qquad - xyz - xz^2 \qquad + y^2z - yz^2 + z^3 \\
\hline
x^3 \qquad\qquad - 3xyz \qquad + y^3 \qquad\qquad + z^3
\end{array}$$

31.

$$\begin{array}{l}
2x^3 - 3x^2 + 5x - 1 \\
3x^3 - x^2 - 2x - 5 \\
\hline
6x^6 - 9x^5 + 15x^4 - 3x^3 \\
\quad - 2x^5 + 3x^4 - 5x^3 + x^2 \\
\qquad - 4x^4 + 6x^3 - 10x^2 + 2x \\
\qquad\qquad - 10x^3 + 15x^2 - 25x + 5 \\
\hline
6x^6 - 11x^5 + 14x^4 - 12x^3 + 6x^2 - 23x + 5
\end{array}$$

32.

$$\begin{array}{l}
ab + cd + ac + bd \\
ab + cd - ac - bd \\
\hline
a^2b^2 + abcd + a^2bc + ab^2d \\
\quad abcd \qquad\qquad + c^2d^2 + ac^2d + bcd^2 \\
\quad - abcd - a^2bc \qquad\qquad - ac^2d \qquad - a^2c^2 \\
\quad - abcd \qquad - ab^2d \qquad\qquad - bcd^2 \qquad - b^2d^2 \\
\hline
a^2b^2 \qquad\qquad + c^2d^2 \qquad\qquad - a^2c^2 - b^2d^2
\end{array}$$

33.

$$\begin{array}{l}
2a^3 - 5a^2 - 6a + 4 \\
4a^3 + 10a^2 - 12a - 8 \\
\hline
8a^6 - 20a^5 - 24a^4 + 16a^3 \\
\qquad 20a^5 - 50a^4 - 60a^3 + 40a^2 \\
\qquad\qquad - 24a^4 + 60a^3 + 72a^2 - 48a \\
\qquad\qquad\qquad - 16a^3 + 40a^2 + 48a - 32 \\
\hline
8a^6 \quad - 98a^4 \quad + 152a^2 \quad - 32
\end{array}$$

34.

$$\begin{array}{l}
x - 3 \\
x + 4 \\
\hline
x^2 - 3x \\
\qquad 4x - 12 \\
\hline
x^2 + x - 12 \\
x - 7 \\
\hline
x^3 + x^2 - 12x \\
\quad - 7x^2 - 7x + 84 \\
\hline
x^3 - 6x^2 - 19x + 84
\end{array}$$

35.

$$\begin{array}{l}
a^2 - ab + b^2 \\
a + b \\
\hline
a^3 - a^2b + ab^2 \\
\qquad a^2b - ab^2 + b^3 \\
\hline
a^3 \qquad\qquad + b^3 \\
a^3 - b^3 \\
\hline
a^6 + a^3b^3 \\
\quad - a^3b^3 - b^6 \\
\hline
a^6 \qquad - b^6
\end{array}$$

36.

$$\begin{array}{l}
2m - 1 \\
3m + 4 \\
\hline
6m^2 - 3m \\
\qquad 8m - 4 \\
\hline
6m^2 + 5m - 4 \\
6m - 5 \\
\hline
36m^3 + 30m^2 - 24m \\
\qquad - 30m^2 - 25m + 20 \\
\hline
36m^3 \qquad - 49m + 20
\end{array}$$

37.

$$\begin{array}{l}
3x - 2 \\
x + 1 \\
\hline
3x^2 - 2x \\
\qquad 3x - 2 \\
\hline
3x^2 + x - 2 \\
3x^2 - x - 2 \\
\hline
9x^4 + 3x^3 - 6x^2 \\
\quad - 3x^3 - x^2 + 2x \\
\qquad - 6x^2 - 2x + 4 \\
\hline
9x^4 \quad - 13x^2 \quad + 4
\end{array}$$

38.

$$\begin{array}{l}
x^2 + x + 1 \\
x^2 - x + 1 \\
\hline
x^4 + x^3 + x^2 \\
\quad - x^3 - x^2 - x \\
\qquad x^2 + x + 1 \\
\hline
x^4 \quad + x^2 \quad + 1 \\
x^4 - x^2 + 1 \\
\hline
x^8 + x^6 + x^4 \\
\quad - x^6 - x^4 - x^2 \\
\qquad x^4 + x^2 + 1 \\
\hline
x^8 \quad + x^4 \quad + 1
\end{array}$$

39.

$$
\begin{array}{l}
a+b \\
a-b \\ \hline
a^2+ab \\
\quad -ab-b^2 \\ \hline
a^2 \qquad -b^2 \\
a^2+b^2 \\ \hline
a^4-a^2b^2 \\
\qquad a^2b^2-b^4 \\ \hline
a^4 \qquad\quad -b^4 \\
a^4+b^4 \\ \hline
a^8-a^4b^4 \\
\qquad a^4b^4-b^8 \\ \hline
a^8 \qquad\quad -b^8
\end{array}
$$

40.

$$
\begin{array}{l}
m+1 \\
m-1 \\ \hline
m^2+m \\
\quad -m-1 \\ \hline
m^2 \qquad -1
\end{array}
\qquad
\begin{array}{l}
m+2 \\
m-2 \\ \hline
m^2+2m \\
\quad -2m-4 \\ \hline
m^2 \qquad -4
\end{array}
\qquad
\begin{array}{l}
m^2-4 \\
m^2-1 \\ \hline
m^4-4m^2 \\
\quad -m^2+4 \\ \hline
m^4-5m^2+4
\end{array}
$$

41.

$$
\begin{array}{l}
2x-1 \\
3x+2 \\ \hline
6x^2-3x \\
\qquad 4x-2 \\ \hline
6x^2+x-2
\end{array}
\qquad
\begin{array}{l}
4x-3 \\
5x+4 \\ \hline
20x^2-15x \\
\qquad 16x-12 \\ \hline
20x^2+x-12
\end{array}
$$

$$
\begin{array}{l}
20x^2+x-12 \\
6x^2+x-2 \\ \hline
120x^4+6x^3-72x^2 \\
\qquad 20x^3+x^2-12x \\
\qquad\qquad -40x^2-2x+24 \\ \hline
120x^4+26x^3-111x^2-14x+24
\end{array}
$$

42.

$$
\begin{array}{l}
a+b \\
a-b \\ \hline
a^2+ab \\
\quad -ab-b^2 \\ \hline
a^2 \qquad -b^2 \\
a+2b \\ \hline
a^3 \qquad -ab^2 \\
\quad 2a^2b \qquad -2b^3 \\ \hline
a^3+2a^2b-ab^2-2b^3 \\
a^3-2a^2b-ab^2+2b^3 \\ \hline
a^6+2a^5b-a^4b^2-2a^3b^3 \\
\quad -2a^5b-4a^4b^2+2a^3b^3+4a^2b^4 \\
\qquad -a^4b^2-2a^3b^3+a^2b^4+2ab^5 \\
\qquad\qquad 2a^3b^3+4a^2b^4-2ab^5-4b^6 \\ \hline
a^6 \qquad -6a^4b^2 \qquad +9a^2b^4 \qquad -4b^6
\end{array}
$$

Art. 83. — Pages 34, 35.

2.

$$\begin{array}{l} a+b+c+d \\ a+b+c+d \\ \hline a^2+ab+ac+ad \\ \quad ab+b^2+bc+bd \\ \quad ac+bc+c^2+cd \\ \quad ad+bd+cd+d^2 \\ \hline a^2+2ab+2ac+2ad+b^2+2bc+2bd+c^2+2cd+d^2 \end{array}$$

3.

$$\begin{array}{l} a-b \\ c-d \\ \hline ac-bc \\ \quad -ad+bd \\ \hline ac-bc-ad+bd \end{array}$$

$$\begin{array}{l} a-c \\ b-d \\ \hline ab-bc \\ \quad -ad+cd \\ \hline ab-bc-ad+cd \end{array}$$

$$\begin{array}{l} ab-ad-bc+cd \\ \quad ac-ad-bc+bd \\ \hline ab+ac-2ad-2bc+bd+cd \end{array}$$

4.

$$\begin{array}{l} 2x-3 \\ 2x-3 \\ \hline 4x^2-6x \\ \quad -6x+9 \\ \hline 4x^2-12x+9 \end{array}$$

$$\begin{array}{l} 3x-9 \\ -x+1 \\ \hline -3x^2+9x \\ \quad 3x-9 \\ \hline -3x^2+12x-9 \\ 4x^2-12x+9 \\ \hline x^2 \end{array}$$

5.

$$\begin{array}{l} a-b+c \\ a-b+c \\ \hline a^2-ab+ac \\ \quad -ab+b^2-bc \\ \quad ac-bc+c^2 \\ \hline a^2-2ab+2ac+b^2-2bc+c^2 \end{array}$$

$$\begin{array}{l} a+b+c \\ a+b+c \\ \hline a^2+ab+ac \\ \quad ab+b^2+bc \\ \quad ac+bc+c^2 \\ \hline a^2+2ab+2ac+b^2+2bc+c^2 \\ a^2-2ab+2ac+b^2-2bc+c^2 \\ \hline 4ab+4bc \end{array}$$

6.

$$\begin{array}{l} a-2b \\ a-3b \\ \hline a^2-2ab \\ \quad -3ab+6b^2 \\ \hline a^2-5ab+6b^2 \\ 4 \\ \hline 4a^2-20ab+24b^2 \end{array}$$

$$\begin{array}{l} 2a-5b \\ 2a-5b \\ \hline 4a^2-10ab \\ \quad -10ab+25b^2 \\ \hline 4a^2-20ab+25b^2 \\ 4a^2-20ab+24b^2 \\ \hline b^2 \end{array}$$

7.

$$\begin{array}{l} a - b \\ a - b \\ \hline a^2 - ab \\ \quad - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array} \qquad \begin{array}{l} a + b \\ a + b \\ \hline a^2 + ab \\ \quad ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

$$\begin{array}{lllll} a^2 & - 2ab & + b^2 & & \\ a^2 & + 2ab & + b^2 & & \\ \hline a^4 & - 2a^3b & + a^2b^2 & & \\ & 2a^3b & - 4a^2b^2 & + 2ab^3 & \\ & & a^2b^2 & - 2ab^3 & + b^4 \\ \hline a^4 & & - 2a^2b^2 & & + b^4 \end{array}$$

8.

$$\begin{array}{lllll} 1 & - x & + x^2 & - x^3 & \\ 1 & + x & & & \\ \hline 1 & - x & + x^2 & - x^3 & \\ & x & - x^2 & + x^3 & - x^4 \\ \hline 1 & & & & - x^4 \\ 1 & + x^4 & & & \\ \hline 1 & - x^4 & & & \\ & x^4 & - x^8 & & \\ \hline 1 & & - x^8 & & \end{array}$$

9.

$$\begin{array}{llll} 1 - a & & & \\ 1 + a^2 & & & \\ \hline 1 - a & & & \\ & & a^2 & - a^3 \\ \hline 1 - a & & + a^2 & - a^3 \end{array}$$

$$\begin{array}{llll} 1 & + a & & \\ 1 & + a & & \\ \hline 1 & + a & & \\ & a & + a^2 & \\ \hline 1 & + 2a & + a^2 & \\ 1 & + a & & \\ \hline 1 & + 2a & + a^2 & \\ & a & + 2a^2 & + a^3 \\ \hline 1 & + 3a & + 3a^2 & + a^3 \\ 1 & - a & + a^2 & - a^3 \\ \hline & 4a & + 2a^2 & + 2a^3 \end{array}$$

10.

$$\begin{array}{llllll} x & - 2y & - 3z & & & \\ x & - 2y & + 3z & & & \\ \hline x^2 & - 2xy & - 3xz & & & \\ & - 2xy & & + 4y^2 & + 6yz & \\ & & 3xz & & - 6yz & - 9z^2 \\ \hline x^2 & - 4xy & & + 4y^2 & & - 9z^2 \end{array}$$

11.

$$x^2 - y(x - y) = x^2 - xy + y^2$$

$$\begin{array}{llll} x^2 & - xy & + y^2 & \\ x & + y & & \\ \hline x^3 & - x^2y & + xy^2 & \\ & x^2y & - xy^2 & + y^3 \\ \hline x^3 & & & + y^3 \end{array}$$

$$\begin{array}{lll} x^3 & + y^3 & \\ x^3 & - y^3 & \\ \hline x^6 & + x^3y^3 & \\ & - x^3y^3 & - y^6 \\ \hline x^6 & & - y^6 \end{array}$$

12.

$$\begin{array}{l} b + c \\ a + b \\ \hline ab + ac + b^2 + bc \end{array}$$

$$\begin{array}{l} c + d \\ a + d \\ \hline ac + ad + cd + d^2 \end{array}$$

$$\begin{array}{l} b - d \\ a + c \\ \hline ab - ad + bc - cd \end{array}$$

$$\begin{array}{lllllll} & ac & + ad & + cd & + d^2 & & \\ ab & & - ad & - cd & & & + bc \\ \hline ab & + ac & & & + d^2 & & + bc \end{array}$$

$$\begin{array}{lllll} ab & + ac & + b^2 & + bc & \\ ab & + ac & & + bc & + d^2 \\ \hline & & b^2 & & - d^2 \end{array}$$

13.

$$\begin{array}{l}
a+b+c \\
a+b+c \\ \hline
a^2+ab+ac \\
\qquad ab+b^2+bc \\
\qquad\qquad ac+bc+c^2 \\ \hline
a^2+2ab+2ac+b^2+2bc+c^2
\end{array}$$

$$\begin{array}{l}
a-b-c \\
a-b-c \\ \hline
a^2-ab-ac \\
\quad -ab+b^2+bc \\
\qquad -ac+bc+c^2 \\ \hline
a^2-2ab-2ac+b^2+2bc+c^2
\end{array}$$

$$\begin{array}{l}
b-c-a \\
b-c-a \\ \hline
b^2-bc-ab \\
\quad -bc+c^2+ac \\
\qquad -ab+ac+a^2 \\ \hline
b^2-2bc-2ab+c^2+2ac+a^2
\end{array}$$

$$\begin{array}{l}
c-a-b \\
c-a-b \\ \hline
c^2-ac-bc \\
\quad -ac+a^2+ab \\
\qquad -bc+ab+b^2 \\ \hline
c^2-2ac-2bc+a^2+2ab+b^2
\end{array}$$

$$\begin{array}{l}
a^2+b^2+c^2+2ab+2bc+2ac \\
a^2+b^2+c^2-2ab+2bc-2ac \\
a^2+b^2+c^2-2ab-2bc+2ac \\
a^2+b^2+c^2+2ab-2bc-2ac \\ \hline
4a^2+4b^2+4c^2
\end{array}$$

14.

$$\begin{array}{l}
b-c \\
a-b \\ \hline
ab-ac-b^2+bc
\end{array}$$

14 (*continued*).

$$\begin{array}{l}
b-c \\
c-a \\ \hline
bc-c^2-ab+ac
\end{array}$$

$$\begin{array}{l}
c-a \\
a-b \\ \hline
ac-a^2-bc+ab
\end{array}$$

$$\begin{array}{l}
\;\;ab+bc-ca\qquad -b^2 \\
-ab+bc+ca\qquad\quad -c^2 \\
\;\;ab-bc+ca-a^2 \\ \hline
ab+bc+ca-a^2-b^2-c^2
\end{array}$$

15.

$$\begin{array}{l}
x-y-z \\
x-y-z \\ \hline
x^2-xy-xz \\
\quad -xy+y^2+yz \\
\qquad -xz+yz+z^2 \\ \hline
x^2-2xy-2xz+y^2+2yz+z^2
\end{array}$$

$$\begin{array}{l}
x-2y \\
x \\ \hline
x^2-2xy
\end{array}
\qquad
\begin{array}{l}
y-2z \\
y \\ \hline
y^2-2yz
\end{array}
\qquad
\begin{array}{l}
z-2x \\
z \\ \hline
z^2-2xz
\end{array}$$

$$\begin{array}{c}
x^2-2xy+y^2-2yz+z^2-2xz \\
x^2-2xy+y^2+2yz+z^2-2xz \\ \hline
-4yz
\end{array}$$

16.

$$\begin{array}{l}
x+1 \\
x+2 \\ \hline
x^2+x \\
\quad 2x+2 \\ \hline
x^2+3x+2 \\
x+3 \\ \hline
x^3+3x^2+2x \\
\quad 3x^2+9x+6 \\ \hline
x^3+6x^2+11x+6 \\
x \\ \hline
x^4+6x^3+11x^2+6x
\end{array}$$

16 (*continued*).

$$\begin{array}{r}
x^4+6x^3+11x^2+6x \\
1 \\
\hline
x^4+6x^3+11x^2+6x+1
\end{array}$$

$$\begin{array}{l}
x^2+3x+1 \\
x^2+3x+1 \\
\hline
x^4+3x^3+x^2 \\
\qquad 3x^3+9x^2+3x \\
\qquad\qquad\quad x^2+3x+1 \\
\hline
x^4+6x^3+11x^2+6x+1
\end{array}$$

$$\begin{array}{c}
x^4+6x^3+11x^2+6x+1 \\
x^4+6x^3+11x^2+6x+1 \\
\hline
0
\end{array}$$

17.

$$\begin{array}{l}
a-b-c \\
a-b-c \\
\hline
a^2-ab-ac \\
\quad -ab \qquad +b^2+bc \\
\qquad\quad -ac \qquad +bc+c^2 \\
\hline
a^2-2ab-2ac+b^2+2bc+c^2
\end{array}$$

$$\begin{array}{l}
c-a-b \\
c-a-b \\
\hline
c^2-ac-bc \\
\quad -ac \qquad +a^2+ab \\
\qquad\quad -bc \qquad +ab+b^2 \\
\hline
c^2-2ac-2bc+a^2+2ab+b^2
\end{array}$$

$$\begin{array}{l}
a^2+b^2+c^2-2ab+2bc-2ac \\
a^2+b^2+c^2+2ab-2bc-2ac \\
\hline
2a^2+2b^2+2c^2 \qquad\qquad -4ac
\end{array}$$

$$\begin{array}{l}
a+b+c \\
a+b+c \\
\hline
a^2+ab+ac \\
\qquad ab \qquad +b^2+bc \\
\qquad\qquad ac \qquad +bc+c^2 \\
\hline
a^2+2ab+2ac+b^2+2bc+c^2
\end{array}$$

17 (*continued*).

$$\begin{array}{l}
b-c-a \\
b-c-a \\
\hline
b^2-bc-ab \\
\quad -bc \qquad +c^2+ac \\
\qquad\quad -ab \qquad +ac+a^2 \\
\hline
b^2-2bc-2ab+c^2+2ac+a^2
\end{array}$$

$$\begin{array}{r}
a^2+b^2+c^2+2ab+2bc+2ac \\
a^2+b^2+c^2-2ab-2bc+2ac \\
\hline
2a^2+2b^2+2c^2 \qquad\qquad +4ac \\
2a^2+2b^2+2c^2 \qquad\qquad -4ac \\
\hline
8ac
\end{array}$$

18.

$$\begin{array}{l}
2m-n \\
2m-n \\
\hline
4m^2-2mn \\
\qquad -2mn+n^2 \\
\hline
4m^2-4mn+n^2
\end{array}$$

$$\begin{array}{r}
m+2n \\
m+2n \\
\hline
m^2+2mn \\
2mn+4n^2 \\
\hline
m^2+4mn+4n^2 \\
4m^2-4mn+n^2 \\
\hline
-3m^2+8mn+3n^2
\end{array}$$

$$\begin{array}{l}
m-2n \\
m-2n \\
\hline
m^2-2mn \\
\quad -2mn+4n^2 \\
\hline
m^2-4mn+4n^2
\end{array}$$

$$\begin{array}{r}
2m+n \\
2m+n \\
\hline
4m^2+2mn \\
2mn+n^2 \\
\hline
4m^2+4mn+n^2 \\
m^2-4mn+4n^2 \\
\hline
3m^2+8mn-3n^2
\end{array}$$

18 (*continued*).

$$
\begin{array}{l}
-3m^2+8mn+3n^2 \\
3m^2+8mn-3n^2 \\
\hline
-9m^4+24m^3n+\ \ 9m^2n^2 \\
\qquad\quad -24m^3n+64m^2n^2+24mn^3 \\
\qquad\qquad\qquad\qquad 9m^2n^2-24mn^3-9n^4 \\
\hline
-9m^4\qquad\qquad +82m^2n^2\qquad\qquad -9n^4
\end{array}
$$

19.

$$
\begin{array}{l}
x+y+z \\
x+y+z \\
\hline
x^2+xy+xz \\
\qquad xy\qquad +y^2+yz \\
\qquad\qquad xz\qquad +yz+z^2 \\
\hline
x^2+2xy+2xz+y^2+2yz+z^2 \\
x+y+z \\
\hline
x^3+2x^2y+2x^2z+xy^2+2xyz+xz^2 \\
\qquad x^2y\qquad +2xy^2+2xyz\qquad +y^3+2y^2z+yz^2 \\
\qquad\qquad x^2z\qquad +2xyz+2xz^2\qquad +y^2z+2yz^2+z^3 \\
\hline
x^3+3x^2y+3x^2z+3xy^2+6xyz+3xz^2+y^3+3y^2z+3yz^2+z^3 \qquad (A)
\end{array}
$$

$$
\begin{array}{l}
y+z \\
z+x \\
\hline
yz+z^2+xy+xz \\
x+y \\
\hline
xyz+xz^2+x^2y+x^2z \\
xyz\qquad\qquad +y^2z+yz^2+xy^2 \\
\hline
2xyz+xz^2+x^2y+x^2z+y^2z+yz^2+xy^2 \\
3 \\
\hline
6xyz+3xz^2+3x^2y+3x^2z+3y^2z+3yz^2+3xy^2 \\
x^3+y^3+z^3 \\
\hline
x^3+y^3+z^3+6xyz+3xz^2+3x^2y+3x^2z+3y^2z+3yz^2+3xy^2 \qquad (B)
\end{array}
$$

Subtracting (B) from (A), there is no remainder.

CHAPTER VI.

Art. 90.—Page 38.

4. $\dfrac{84}{-12} = -7.$

5. $\dfrac{-343}{7} = -49.$

6. $\dfrac{-324}{-18} = 18.$

7. $\dfrac{444}{-37} = -12.$

8. $\dfrac{12a^5}{4a} = 3a^4.$

9. $\dfrac{-a^2c}{ac} = -a.$

10. $\dfrac{2m^3n^4}{-mn^3} = -2m^2n.$

11. $\dfrac{-8x^2y^2}{-4x^2} = 2y^2.$

12. $\dfrac{30a^5b^3}{5a^2b} = 6a^3b^2.$

13. $\dfrac{14m^3n^4}{-7mn^3} = -2m^2n.$

14. $\dfrac{-18x^2y^5z}{9x^2z} = -2y^5.$

15. $\dfrac{-65a^3b^3c^3}{-5ab^2c^3} = 13a^2b.$

16. $\dfrac{72m^5n}{-12m^2} = -6m^3n.$

17. $\dfrac{12x^ay^b}{3x^cy^d} = 4x^{a-c}y^{b-d}.$

18. $\dfrac{-18a^mb}{6ab} = -3a^{m-1}.$

19. $\dfrac{-144c^5d^7e^6}{-36c^2d^3e} = 4c^3d^4e^5.$

20. $\dfrac{-3a^{m+2}}{a^{m+1}} = -3a.$

21. $\dfrac{a^{m+n}b^{m+n}}{a^mb^n} = a^nb^m.$

22. $\dfrac{-91x^4y^3z^2}{-13x^3y^2} = 7xyz^2.$

23. $\dfrac{18m^3n^4p^5}{-2m^2np^5} = -9mn^3.$

Art. 91.—Pages 38, 39.

2. $\dfrac{8a^3bc + 16a^5bc - 4a^2c^2}{4a^2c} = 2ab + 4a^3b - c.$

3. $\dfrac{9x^4 + 27x^3 - 21x^2}{-3x^2} = -3x^2 - 9x + 7.$

4. $\dfrac{30a^3 - 75a^4b}{15a^3} = 2 - 5ab.$

5. $\dfrac{2x^3y^2z - 12xy^2z^3}{-2xy^2z} = -x^2 + 6z^2.$

6. $\dfrac{5a^2bc - 5ab^2c + 5abc^2}{-5abc} = -a + b - c.$

7. $\dfrac{4x^7 - 8x^6 - 14x^5 + 2x^4 - 6x^3}{2x^3} = 2x^4 - 4x^3 - 7x^2 + x - 3.$

8. $\dfrac{-12a^pb^q - 30a^{12}b^3 + 108a^nb^n}{-6a^mb^n} = 2a^{p-m}b^{q-n} + 5a^{12-m}b^{3-n} - 18a^{n-m}.$

9. $\dfrac{20x^4 - 12x^2 - 28x}{4x} = 5x^3 - 3x - 7.$

10. $\dfrac{-a^2b^2c - ab^2c^2 + a^2bc^2}{-abc} = ab + bc - ac.$

11. $\dfrac{9a^5bc - 3a^2b + 18a^3bc}{-3ab} = -3a^4c + a - 6a^2c.$

12. $\dfrac{15x^my^nz^r - 35x^{m+2}y^{2n}z}{5x^my^nz} = 3z^{r-1} - 7x^2y^n.$

13. $\dfrac{20a^4bc + 15abd^3 - 10a^2b}{-5ab} = -4a^3c - 3d^3 + 2a.$

Art. 93.—Pages 41, 42, 43.

4.

$$\begin{array}{l|l} 6x^2 - x - 35 & 3x + 7 \\ \cline{2-2} 6x^2 + 14x & 2x - 5 \\ \hline \quad -15x - 35 & \\ \quad -15x - 35 & \\ \hline \end{array}$$

5.

$$\begin{array}{l|l} 2 - 3ax - 2a^2x^2 & 1 - 2ax \\ \cline{2-2} 2 - 4ax & 2 + ax \\ \hline \quad ax - 2a^2x^2 & \\ \quad ax - 2a^2x^2 & \\ \hline \end{array}$$

6.

$$\begin{array}{l|l} a^2 - 4ab + 4b^2 & a - 2b \\ \cline{2-2} a^2 - 2ab & a - 2b \\ \hline \quad -2ab + 4b^2 & \\ \quad -2ab + 4b^2 & \\ \hline \end{array}$$

7.

$$\begin{array}{l|l} -56 + 59x - 15x^2 & -7 + 3x \\ \cline{2-2} -56 + 24x & 8 - 5x \\ \hline \quad 35x - 15x^2 & \\ \quad 35x - 15x^2 & \\ \hline \end{array}$$

8.

$$\begin{array}{l|l} 3b^3 + 3ab^2 - 4a^2b - 4a^3 & b + a \\ \cline{2-2} 3b^3 + 3ab^2 & 3b^2 - 4a^2 \\ \hline \quad -4a^2b - 4a^3 & \\ \quad -4a^2b - 4a^3 & \\ \hline \end{array}$$

9.

$$\begin{array}{l|l} -2ax^3 + 2a^3x & ax - a^2 \\ \cline{2-2} -2ax^3 + 2a^2x^2 & -2x^2 - 2ax \\ \hline \quad -2a^2x^2 + 2a^3x & \\ \quad -2a^2x^2 + 2a^3x & \\ \hline \end{array}$$

10.

$$\begin{array}{l|l} 18x^3-5x+1 & 6x^2+2x-1 \\ 18x^3+6x^2-3x & 3x-1 \\ \hline -6x^2-2x+1 & \\ -6x^2-2x+1 & \\ \hline \end{array}$$

11.

$$\begin{array}{l|l} 8m^3-36m+35 & 2m+5 \\ 8m^3+20m^2 & 4m^2-10m+7 \\ \hline -20m^2-36m+35 & \\ -20m^2-50m & \\ \hline 14m+35 & \\ 14m+35 & \\ \hline \end{array}$$

12.

$$\begin{array}{l|l} 27x^3+y^3 & 3x+y \\ 27x^3+9x^2y & 9x^2-3xy+y^2 \\ \hline -9x^2y+y^3 & \\ -9x^2y-3xy^2 & \\ \hline 3xy^2+y^3 & \\ 3xy^2+y^3 & \\ \hline \end{array}$$

13.

$$\begin{array}{l|l} 16m^4-1 & 2m-1 \\ 16m^4-8m^3 & 8m^3+4m^2+2m+1 \\ \hline 8m^3-1 & \\ 8m^3-4m^2 & \\ \hline 4m^2-1 & \\ 4m^2-2m & \\ \hline 2m-1 & \\ 2m-1 & \\ \hline \end{array}$$

14.

$$\begin{array}{l|l} a^2-2ac-b^2+c^2 & a+b-c \\ a^2+ab-ac & a-b-c \\ \hline -ab-ac-b^2+c^2 & \\ -ab-b^2+bc & \\ \hline -ac-bc+c^2 & \\ -ac-bc+c^2 & \\ \hline \end{array}$$

15.

$$\begin{array}{l|l} 8a^3+36a^2b+54ab^2+27b^3 & 2a+3b \\ 8a^3+12a^2b & 4a^2+12ab+9b^2 \\ \hline 24a^2b+54ab^2+27b^3 & \\ 24a^2b+36ab^2 & \\ \hline 18ab^2+27b^3 & \\ 18ab^2+27b^3 & \\ \hline \end{array}$$

16.

$$\begin{array}{l|l} x^4+x^2y^2+y^4 & x^2+xy+y^2 \\ x^4+x^3y+x^2y^2 & x^2-xy+y^2 \\ \hline -x^3y+y^4 & \\ -x^3y-x^2y^2-xy^3 & \\ \hline x^2y^2+xy^3+y^4 & \\ x^2y^2+xy^3+y^4 & \\ \hline \end{array}$$

17.

$$\begin{array}{l|l} 2x^4-19x^2+9 & 2x^3+6x^2-x-3 \\ 2x^4+6x^3-x^2-3x & x-3 \\ \hline -6x^3-18x^2+3x+9 & \\ -6x^3-18x^2+3x+9 & \\ \hline \end{array}$$

18.

$$\begin{array}{l|l} 8m^3-4m^2n-6mn^2+3n^3 & 2m-n \\ 8m^3-4m^2n & 4m^2-3n^2 \\ \hline -6mn^2+3n^3 & \\ -6mn^2+3n^3 & \\ \hline \end{array}$$

19.

$$
\begin{array}{l|l}
4x^4-8x^3-6x^2+24 & 2x-4 \\ \cline{2-2}
\underline{4x^4-8x^3} & 2x^3-3x-6 \\
\quad -6x^2+24 & \\
\quad \underline{-6x^2+12x} & \\
\qquad -12x+24 & \\
\qquad \underline{-12x+24} &
\end{array}
$$

20.

$$
\begin{array}{l|l}
6x^4-31x^3+23x^2-\ \ 2x-48 & 3x^2-5x+6 \\ \cline{2-2}
\underline{6x^4-10x^3+12x^2} & 2x^2-7x-8 \\
\quad -21x^3+11x^2-\ \ 2x-48 & \\
\quad \underline{-21x^3+35x^2-42x} & \\
\qquad -24x^2+40x-48 & \\
\qquad \underline{-24x^2+40x-48} &
\end{array}
$$

21.

$$
\begin{array}{l|l}
4a^5-\ \ a^3+27 & 2a^4-3a^3+4a^2-6a+9 \\ \cline{2-2}
\underline{4a^5-6a^4+\ \ 8a^3-12a^2+18a} & 2a+3 \\
\quad 6a^4-\ \ 9a^3+12a^2-18a+27 & \\
\quad \underline{6a^4-\ \ 9a^3+12a^2-18a+27} &
\end{array}
$$

22.

$$
\begin{array}{l|l}
x^4-9x^2-6xy-y^2 & x^2+3x+y \\ \cline{2-2}
\underline{x^4+3x^3+\ \ x^2y} & x^2-3x-y \\
\quad -3x^3-9x^2-x^2y-6xy-y^2 & \\
\quad \underline{-3x^3-9x^2\qquad\quad -3xy} & \\
\qquad -x^2y-3xy-y^2 & \\
\qquad \underline{-x^2y-3xy-y^2} &
\end{array}
$$

23.

$$
\begin{array}{l|l}
a^8-81b^4 & a^2+3b \\ \cline{2-2}
\underline{a^8+\ \ 3a^6b} & a^6-\ \ 3a^4b+9a^2b^2-27b^3 \\
\quad -\ \ 3a^6b-81b^4 & \\
\quad \underline{-\ \ 3a^6b-\ \ 9a^4b^2} & \\
\qquad 9a^4b^2-81b^4 & \\
\qquad \underline{9a^4b^2+27a^2b^3} & \\
\qquad\quad -27a^2b^3-81b^4 & \\
\qquad\quad \underline{-27a^2b^3-81b^4} &
\end{array}
$$

24.

$$
\begin{array}{l|l}
x^2 - y^2 + 2yz - z^2 & x + y - z \\ \cline{2-2}
x^2 + xy - xz & x - y + z \\ \hline
-xy + xz - y^2 + 2yz - z^2 & \\
-xy \qquad - y^2 + yz & \\ \cline{1-1}
xz + yz - z^2 & \\
xz + yz - z^2 & \\ \cline{1-1}
\end{array}
$$

25.

$$
\begin{array}{l|l}
3x^4 - 14x^2 + 8 & x - 2 \\ \cline{2-2}
3x^4 - 6x^3 & 3x^3 + 6x^2 - 2x - 4 \\ \hline
6x^3 - 14x^2 + 8 & \\
6x^3 - 12x^2 & \\ \cline{1-1}
- 2x^2 + 8 & \\
- 2x^2 + 4x & \\ \cline{1-1}
-4x + 8 & \\
-4x + 8 & \\ \cline{1-1}
\end{array}
$$

26.

$$
\begin{array}{l|l}
x^5y + y^6 & x + y \\ \cline{2-2}
x^5y + x^4y^2 & x^4y - x^3y^2 + x^2y^3 - xy^4 + y^5 \\ \hline
- x^4y^2 + y^6 & \quad = y\,(x^4 - x^3y + x^2y^2 - xy^3 + y^4) \\
- x^4y^2 - x^3y^3 & \\ \cline{1-1}
x^3y^3 + y^6 & \\
x^3y^3 + x^2y^4 & \\ \cline{1-1}
- x^2y^4 + y^6 & \\
- x^2y^4 - xy^5 & \\ \cline{1-1}
xy^5 + y^6 & \\
xy^5 + y^6 & \\ \cline{1-1}
\end{array}
$$

27.

$$
\begin{array}{l|l}
15m^4 - 32m^3 + 50m^2 - 32m + 15 & 3m^2 - 4m + 5 \\ \cline{2-2}
15m^4 - 20m^3 + 25m^2 & 5m^2 - 4m + 3 \\ \hline
- 12m^3 + 25m^2 - 32m + 15 & \\
- 12m^3 + 16m^2 - 20m & \\ \cline{1-1}
9m^2 - 12m + 15 & \\
9m^2 - 12m + 15 & \\ \cline{1-1}
\end{array}
$$

28.

$$
\begin{array}{l|l}
3x^4+4x^3+1 & x^2+2x+1 \\
\underline{3x^4+6x^3+3x^2} & 3x^2-2x+1 \\
\quad -2x^3-3x^2+1 & \\
\quad \underline{-2x^3-4x^2-2x} & \\
\qquad\qquad x^2+2x+1 & \\
\qquad\qquad \underline{x^2+2x+1} &
\end{array}
\qquad
\begin{array}{r}
x+1 \\
\underline{x+1} \\
x^2+x \quad \\
\underline{x+1} \\
x^2+2x+1
\end{array}
$$

29.

$$
\begin{array}{l|l}
21a^5-21b^5 & 7a-7b \\
\underline{21a^5-21a^4b} & 3a^4+3a^3b+3a^2b^2+3ab^3+3b^4 \\
\quad 21a^4b-21b^5 & \quad =3(a^4+a^3b+a^2b^2+ab^3+b^4) \\
\quad \underline{21a^4b-21a^3b^2} & \\
\qquad 21a^3b^2-21b^5 & \\
\qquad \underline{21a^3b^2-21a^2b^3} & \\
\qquad\quad 21a^2b^3-21b^5 & \\
\qquad\quad \underline{21a^2b^3-21ab^4} & \\
\qquad\qquad 21ab^4-21b^5 & \\
\qquad\qquad \underline{21ab^4-21b^5} &
\end{array}
$$

30.

$$
\begin{array}{l|l}
64x^4+1 & 8x^2-4x+1 \\
\underline{64x^4-32x^3+8x^2} & 8x^2+4x+1 \\
\quad 32x^3-8x^2+1 & \\
\quad \underline{32x^3-16x^2+4x} & \\
\qquad 8x^2-4x+1 & \\
\qquad \underline{8x^2-4x+1} &
\end{array}
$$

31.

$$
\begin{array}{l|l}
9x^4-67x^2+50x+24 & x^2+x-6 \\
\underline{9x^4+9x^3-54x^2} & 9x^2-9x-4 \\
\quad -9x^3-13x^2+50x+24 & \\
\quad \underline{-9x^3-9x^2+54x} & \\
\qquad -4x^2-4x+24 & \\
\qquad \underline{-4x^2-4x+24} &
\end{array}
$$

32.

$$\begin{array}{l|l} x^4-4x^3y+6x^2y^2-4xy^3+y^4 & x^2-2xy+y^2 \\ \underline{x^4-2x^3y+\ \ x^2y^2} & x^2-2xy+y^2 \\ \quad -2x^3y+5x^2y^2-4xy^3+y^4 & \\ \quad \underline{-2x^3y+4x^2y^2-2xy^3} & \\ \qquad x^2y^2-2xy^3+y^4 & \\ \qquad \underline{x^2y^2-2xy^3+y^4} & \end{array}$$

33.

$$\begin{array}{l|l} x^4-4x^3+2x^2+4x+1 & x^2-2x-1 \\ \underline{x^4-2x^3-\ \ x^2} & x^2-2x-1 \\ \quad -2x^3+3x^2+4x+1 & \\ \quad \underline{-2x^3+4x^2+2x} & \\ \qquad -\ \ x^2+2x+1 & \\ \qquad \underline{-\ \ x^2+2x+1} & \end{array}$$

$$\begin{array}{r} x-1 \\ \underline{x-1} \\ x^2-\ \ x\qquad \\ \underline{-\ \ x+1} \\ x^2-2x+1 \\ \underline{2} \\ x^2-2x-1 \end{array}$$

34.

$$\begin{array}{l|l} 9x^4-37x^2y^2+\ \ 4y^4 & 3x^2+5xy-2y^2 \\ \underline{9x^4+15x^3y\ -\ \ 6x^2y^2} & 3x^2-5xy-2y^2 \\ \quad -15x^3y\ -31x^2y^2+\ \ 4y^4 & \\ \quad \underline{-15x^3y\ -25x^2y^2+10xy^3} & \\ \qquad -\ \ 6x^2y^2-10xy^3+4y^4 & \\ \qquad \underline{-\ \ 6x^2y^2-10xy^3+4y^4} & \end{array}$$

35.

$$\begin{array}{l|l} a^4+\ \ a^2b^2+25b^4 & a^2-3ab+5b^2 \\ \underline{a^4-3a^3b\ +\ \ 5a^2b^2} & a^2+3ab+5b^2 \\ \quad 3a^3b\ -\ \ 4a^2b^2+25b^4 & \\ \quad \underline{3a^3b\ -\ \ 9a^2b^2+15ab^3} & \\ \qquad 5a^2b^2-15ab^3+25b^4 & \\ \qquad \underline{5a^2b^2-15ab^3+25b^4} & \end{array}$$

$$\begin{array}{r} a-5b \\ \underline{a-\ \ b} \\ a^2-5ab\qquad \\ \underline{-\ \ ab+5b^2} \\ a^2-6ab+5b^2 \\ \underline{3ab\qquad} \\ a^2-3ab+5b^2 \end{array}$$

36.

$$\begin{array}{l|l} 6x^5-11x^3+3x^2+4x-4 & 3x^2-4 \\ \underline{6x^5-\ \ 8x^3} & 2x^3-x+1 \\ \quad -\ \ 3x^3+3x^2+4x-4 & \\ \quad \underline{-\ \ 3x^3\qquad +4x} & \\ \qquad 3x^2\qquad -4 & \\ \qquad \underline{3x^2\qquad -4} & \end{array}$$

37.

$$\begin{array}{l|l}
6x^5+15x^3+51x-18 & 2x^3-4x^2+7x-2 \\
\cline{2-2}
6x^5-12x^4+21x^3-6x^2 & 3x^2+6x+9 \\
\hline
\quad 12x^4-6x^3+6x^2+51x-18 & \\
\quad 12x^4-24x^3+42x^2-12x & \\
\cline{1-1}
\qquad 18x^3-36x^2+63x-18 & \\
\qquad 18x^3-36x^2+63x-18 & \\
\cline{1-1}
\end{array}$$

38.

$$\begin{array}{l|l}
2x^4-3x^3-4x^2-11x-12 & 2x^2+x+4 \\
\cline{2-2}
2x^4+x^3+4x^2 & x^2-2x-3 \\
\hline
\quad -4x^3-8x^2-11x-12 & \\
\quad -4x^3-2x^2-8x & \\
\cline{1-1}
\qquad -6x^2-3x-12 & \\
\qquad -6x^2-3x-12 & \\
\cline{1-1}
\end{array}$$

39.

$$\begin{array}{l|l}
m^5-17m^3+12m^2+52m-48 & m^2+m-2 \\
\cline{2-2}
m^5+m^4-2m^3 & m^3-m^2-14m+24 \\
\hline
\quad -m^4-15m^3+12m^2+52m-48 & \\
\quad -m^4-m^3+2m^2 & \\
\cline{1-1}
\qquad -14m^3+10m^2+52m-48 & \\
\qquad -14m^3-14m^2+28m & \\
\cline{1-1}
\qquad\quad 24m^2+24m-48 & \\
\qquad\quad 24m^2+24m-48 & \\
\cline{1-1}
\end{array}$$

40.

$$\begin{array}{l|l}
x^{n+1}+x^ny-xy^n-y^{n+1} & x^n-y^n \\
\cline{2-2}
x^{n+1}-xy^n & x+y \\
\hline
\quad x^ny-y^{n+1} & \\
\quad x^ny-y^{n+1} & \\
\cline{1-1}
\end{array}$$

41.

$$\begin{array}{l|l}
x^5y-xy^5 & x^3+x^2y+xy^2+y^3 \\
\cline{2-2}
x^5y+x^4y^2+x^3y^3+x^2y^4 & x^2y-xy^2 \\
\hline
\quad -x^4y^2-x^3y^3-x^2y^4-xy^5 & \\
\quad -x^4y^2-x^3y^3-x^2y^4-xy^5 & \\
\cline{1-1}
\end{array}$$

42.

$$
\begin{array}{l|l}
x^5 - 6x^2 - x - 6 & x^2 + 2x + 3 \\ \cline{2-2}
x^5 + 2x^4 + 3x^3 & x^3 - 2x^2 + x - 2 \\ \hline
\quad -2x^4 - 3x^3 - 6x^2 - x - 6 & \\
\quad -2x^4 - 4x^3 - 6x^2 & \\ \cline{1-1}
\qquad\qquad x^3 \qquad - x - 6 & \\
\qquad\qquad x^3 + 2x^2 + 3x & \\ \cline{1-1}
\qquad\qquad\quad -2x^2 - 4x - 6 & \\
\qquad\qquad\quad -2x^2 - 4x - 6 & \\ \cline{1-1}
\end{array}
$$

43.

$$
\begin{array}{l|l}
2a^5 - 9a^4b - 7a^3b^2 + 53a^2b^3 - 49b^5 & 2a^2 - 5ab - 7b^2 \\ \cline{2-2}
2a^5 - 5a^4b - 7a^3b^2 & a^3 - 2a^2b - 5ab^2 + 7b^3 \\ \hline
\quad -4a^4b \qquad\qquad + 53a^2b^3 - 49b^5 & \\
\quad -4a^4b + 10a^3b^2 + 14a^2b^3 & \\ \cline{1-1}
\qquad\qquad -10a^3b^2 + 39a^2b^3 - 49b^5 & \\
\qquad\qquad -10a^3b^2 + 25a^2b^3 + 35ab^4 & \\ \cline{1-1}
\qquad\qquad\qquad\quad 14a^2b^3 - 35ab^4 - 49b^5 & \\
\qquad\qquad\qquad\quad 14a^2b^3 - 35ab^4 - 49b^5 & \\ \cline{1-1}
\end{array}
$$

44.

$$
\begin{array}{l|l}
x^6 - 6x^4 + 5x^2 - 1 & x^3 + 2x^2 - x - 1 \\ \cline{2-2}
x^6 + 2x^5 - x^4 - x^3 & x^3 - 2x^2 - x + 1 \\ \hline
\quad -2x^5 - 5x^4 + x^3 + 5x^2 - 1 & \\
\quad -2x^5 - 4x^4 + 2x^3 + 2x^2 & \\ \cline{1-1}
\qquad\qquad - x^4 - x^3 + 3x^2 - 1 & \\
\qquad\qquad - x^4 - 2x^3 + x^2 + x & \\ \cline{1-1}
\qquad\qquad\qquad\quad x^3 + 2x^2 - x - 1 & \\
\qquad\qquad\qquad\quad x^3 + 2x^2 - x - 1 & \\ \cline{1-1}
\end{array}
$$

45.

$$
\begin{array}{l|l}
2x^2 + xy - 2xz - 6y^2 + 17yz - 12z^2 & 2x - 3y + 4z \\ \cline{2-2}
2x^2 - 3xy + 4xz & x + 2y - 3z \\ \hline
\qquad 4xy - 6xz - 6y^2 + 17yz - 12z^2 & \\
\qquad 4xy \qquad\quad - 6y^2 + 8yz & \\ \cline{1-1}
\qquad\qquad -6xz \qquad\quad + 9yz - 12z^2 & \\
\qquad\qquad -6xz \qquad\quad + 9yz - 12z^2 & \\ \cline{1-1}
\end{array}
$$

46.

$$\begin{array}{l|l}
a^{2n} - b^{2m} + 2b^mc^r - c^{2r} & a^n + b^m - c^r \\
\underline{a^{2n} + a^nb^m - a^nc^r} & a^n - b^m + c^r \\
\quad - a^nb^m + a^nc^r - b^{2m} + 2b^mc^r - c^{2r} & \\
\quad \underline{- a^nb^m \qquad\quad - b^{2m} + \ \ b^mc^r} & \\
\qquad\qquad a^nc^r \qquad + b^mc^r - c^{2r} & \\
\qquad\qquad \underline{a^nc^r \qquad + b^mc^r - c^{2r}} &
\end{array}$$

47.

$$\begin{array}{l|l}
x^6 - 6x^4 - 3x^2 - 1 & x^3 - 2x^2 - x - 1 \\
\underline{x^6 - 2x^5 - \ x^4 - \ x^3} & x^3 + 2x^2 - x + 1 \\
\quad 2x^5 - 5x^4 + \ x^3 - 3x^2 - 1 & \\
\quad \underline{2x^5 - 4x^4 - 2x^3 - 2x^2} & \\
\qquad - x^4 + 3x^3 - \ x^2 - 1 & \\
\qquad \underline{- x^4 + 2x^3 + \ x^2 + \ x} & \\
\qquad\qquad x^3 - 2x^2 - \ x - 1 & \\
\qquad\qquad \underline{x^3 - 2x^2 - \ x - 1} &
\end{array}$$

48.

$$\begin{array}{l|l}
12a^5 - 14a^4b + 10a^3b^2 - \ a^2b^3 - 8ab^4 + 4b^5 & 6a^3 - 4a^2b - 3ab^2 + 2b^3 \\
\underline{12a^5 - \ 8a^4b - \ 6a^3b^2 + 4a^2b^3} & 2a^2 - \ ab + 2b^2 \\
\quad - 6a^4b + 16a^3b^2 - 5a^2b^3 - 8ab^4 + 4b^5 & \\
\quad \underline{- 6a^4b + \ 4a^3b^2 + 3a^2b^3 - 2ab^4} & \\
\qquad\qquad 12a^3b^2 - 8a^2b^3 - 6ab^4 + 4b^5 & \\
\qquad\qquad \underline{12a^3b^2 - 8a^2b^3 - 6ab^4 + 4b^5} &
\end{array}$$

50.

$$\begin{array}{l|l}
x^3 + (a + b + c)\,x^2 + (ab + bc + ca)\,x + abc & x^2 + (b + c)\,x + bc \\
\underline{x^3 + (\quad b + c)\,x^2 \qquad\quad + bcx} & x + a \\
\qquad\quad ax^2 + (ab + ca)\,x + abc & \\
\qquad\quad \underline{ax^2 + (ab + ca)\,x + abc} &
\end{array}$$

51.

$$\begin{array}{l|l}
(b + c)\,a^2 + (b^2 + 3bc + c^2)\,a + bc\,(b + c) & a + b + c \\
\underline{(b + c)\,a^2 + (b^2 + 2bc + c^2)\,a} & a\,(b + c) + bc \\
\qquad\qquad bca \qquad + bc\,(b + c) & \\
\qquad\qquad \underline{bca \qquad + bc\,(b + c)} &
\end{array}$$

52.

$$
\begin{array}{l|l}
(x+y)^2-5(x+y)+6 & (x+y)-2 \\
\underline{(x+y)^2-2(x+y)\qquad\qquad} & (x+y)-3 \\
\qquad\quad -3(x+y)+6 & \\
\qquad\quad \underline{-3(x+y)+6} &
\end{array}
$$

53.

$$
\begin{array}{l|l}
(a+b)^3+1 & (a+b)+1 \\
\underline{(a+b)^3+(a+b)^2} & (a+b)^2-(a+b)+1 \\
\qquad -(a+b)^2+1 & \\
\qquad \underline{-(a+b)^2-(a+b)} & \\
\qquad\qquad\qquad (a+b)+1 & \\
\qquad\qquad\qquad \underline{(a+b)+1} &
\end{array}
$$

54.

$$
\begin{array}{l|l}
x^3+(a+b-c)x^2+(ab-bc-ca)x-abc & x^2+(b-c)x-bc \\
\underline{x^3+(\quad b-c)x^2 \qquad\quad -bcx \qquad\qquad} & x+a \\
\qquad\qquad ax^2+(ab-ca)x \qquad -abc & \\
\qquad\qquad \underline{ax^2+(ab-ca)x \qquad -abc} &
\end{array}
$$

55.

$$
\begin{array}{l|l}
(m-n)^4-2(m-n)^2+1 & (m-n)^2-2(m-n)+1 \\
\underline{(m-n)^4-2(m-n)^3+(m-n)^2} & (m-n)^2+2(m-n)+1 \\
\qquad 2(m-n)^3-3(m-n)^2+1 & \\
\qquad \underline{2(m-n)^3-4(m-n)^2+2(m-n)} & \\
\qquad\qquad\qquad (m-n)^2-2(m-n)+1 & \\
\qquad\qquad\qquad \underline{(m-n)^2-2(m-n)+1} &
\end{array}
$$

56.

$$
\begin{array}{l|l}
x^3+(a-b+c)x^2+(ac-ab-bc)x-abc & x+c \\
\underline{x^3 \qquad\quad + \; cx^2 \qquad\qquad\qquad\qquad\qquad} & x^2+(a-b)x-ab \\
\qquad (a-b)x^2+(ac-ab-bc)x-abc & \\
\qquad \underline{(a-b)x^2+(ac \qquad -bc)x \qquad\;} & \\
\qquad\qquad\qquad -abx \qquad -abc & \\
\qquad\qquad\qquad \underline{-abx \qquad -abc} &
\end{array}
$$

57.

$$
\begin{array}{l|l}
x^4+(3-b)x^3+(c-3b-2)x^2+(2b+3c)x-2c & x^2+3x-2 \\
\underline{x^4+3x^3 \qquad\qquad -2x^2\qquad\qquad\qquad\qquad} & \overline{x^2-bx+c} \\
\qquad -bx^3+(c-3b)x^2 \quad +(2b+3c)x-2c & \\
\qquad \underline{-bx^3 \quad -3bx^2 \qquad +2bx \qquad\qquad} & \\
\qquad\qquad cx^2 \qquad\qquad +3cx \quad -2c & \\
\qquad\qquad \underline{cx^2 \qquad\qquad +3cx \quad -2c} &
\end{array}
$$

58.

$$
\begin{array}{l|l}
a^2(b+c)+a(b^2+\quad bc+c^2)-bc(b+c) & a+b+c \\
\underline{a^2(b+c)+a(b^2+2bc+c^2)\qquad\qquad\quad} & \overline{a(b+c)-bc} \\
\qquad\qquad -abc \qquad -bc(b+c) & \\
\qquad\qquad \underline{-abc \qquad -bc(b+c)} &
\end{array}
$$

CHAPTER VII.

Art. 96.—Pages 45, 46.

4. $(x-4)^2 = x^2 - 8x + 16.$

5. $(3+a)^2 = 9 + 6a + a^2.$

6. $(x+3)(x-3) = x^2 - 9.$

7. $(3a+5)^2 = 9a^2 + 30a + 25.$

8. $(2x+1)(2x-1) = 4x^2 - 1.$

9. $(7-2x)^2 = 49 - 28x + 4x^2.$

10. $(2m+3n)^2 = 4m^2 + 12mn + 9n^2.$

11. $(4ab-x)^2 = 16a^2b^2 - 8abx + x^2.$

12. $(5+7x)(5-7x) = 25 - 49x^2.$

13. $(x^4-y^2)^2 = x^8 - 2x^4y^2 + y^4.$

14. $(3x+11)(3x-11) = 9x^2 - 121.$

15. $(x^2y+4)^2 = x^4y^2 + 8x^2y + 16.$

16. $(3x^3+13)^2 = 9x^6 + 78x^3 + 169.$

17. $(6a^2-b^2c)^2 = 36a^4 - 12a^2b^2c + b^4c^2.$

18. $(5a+7b^2)(5a-7b^2) = 25a^2 - 49b^4.$

19. $(13ab+5ac)^2 = 169a^2b^2 + 130a^2bc + 25a^2c^2.$

20. $(x^3+5x)(x^3-5x) = x^6 - 25x^2.$

21. $(1-12xyz)^2 = 1 - 24xyz + 144x^2y^2z^2.$

22. $(4x^2+3y^5)(4x^2-3y^5) = 16x^4 - 9y^{10}.$

23. $(10x^3+9x^2)^2 = 100x^6 + 180x^5 + 81x^4.$

24. $(4a^p-5b^q)^2 = 16a^{2p} - 40a^pb^q + 25b^{2q}.$

25. $(a^m+a^n)(a^m-a^n) = a^{2m} - a^{2n}.$

26. $(7x^3+11x)^2 = 49x^6 + 154x^4 + 121x^2.$

27. $(5a^m-a^n)^2 = 25a^{2m} - 10a^{m+n} + a^{2n}.$

30. $(x+y+z)(x-y+z) = [(x+z)+y][(x+z)-y]$
$= (x+z)^2 - y^2 = x^2 - y^2 + 2xz + z^2.$

31. $(x+y+z)(x-y-z) = [x+(y+z)][x-(y+z)]$
$= x^2 - (y+z)^2 = x^2 - y^2 - 2yz - z^2.$

32. $(1+a-b)(1-a+b) = [1+(a-b)][1-(a-b)]$
$= 1-(a-b)^2 = 1 - a^2 + 2ab - b^2.$

33. $(x^2+x+1)(x^2-x-1)=[x^2+(x+1)][x^2-(x+1)]$
$=x^4-(x+1)^2=x^4-x^2-2x-1.$

34. $(a+b-c)(a-b-c)=[(a-c)+b][(a-c)-b]$
$=(a-c)^2-b^2=a^2-b^2-2ac+c^2.$

35. $(a^2+2a+1)(a^2-2a+1)=[(a^2+1)+2a][(a^2+1)-2a]$
$=(a^2+1)^2-4a^2=a^4+2a^2+1-4a^2=a^4-2a^2+1.$

36. $(x^2+2x-3)(x^2-2x+3)=[x^2+(2x-3)][x^2-(2x-3)]$
$=x^4-(2x-3)^2=x^4-4x^2+12x-9.$

37. $(m^2+mn+n^2)(m^2-mn+n^2)=[(m^2+n^2)+mn][(m^2+n^2)-mn]$
$=(m^2+n^2)^2-m^2n^2=m^4+2m^2n^2+n^4-m^2n^2=m^4+m^2n^2+n^4.$

Art. 97.—Page 47.

2. $(x+7)(x+5)=x^2+12x+35.$
3. $(x-3)(x-4)=x^2-7x+12.$
4. $(x+8)(x-2)=x^2+6x-16.$
5. $(x-3)(x+1)=x^2-2x-3.$
6. $(x-5)(x+6)=x^2+x-30.$
7. $(x+1)(x+12)=x^2+13x+12.$
8. $(x-7)(x+2)=x^2-5x-14.$
9. $(x-8)(x-6)=x^2-14x+48.$
10. $(x+9)(x-5)=x^2+4x-45.$
11. $(x-8)(x-9)=x^2-17x+72.$
12. $(x+4m)(x+6m)=x^2+10mx+24m^2.$
13. $(x-5a)(x+a)=x^2-4ax-5a^2.$
14. $(a+b)(a-4b)=a^2-3ab-4b^2.$
15. $(a+5b)(a+8b)=a^2+13ab+40b^2.$
16. $(x^2-3)(x^2-7)=x^4-10x^2+21.$
17. $(x^3+2a)(x^3-6a)=x^6-4ax^3-12a^2.$

Art. 98.—Pages 48, 49.

4. $\frac{x^2-81}{x-9}=x+9.$
5. $\frac{25-16a^2}{5+4a}=5-4a.$
6. $\frac{x^3+1}{x+1}=x^2-x+1.$
7. $\frac{1-m^3}{1-m}=1+m+m^2.$
8. $\frac{a^3-8}{a-2}=a^2+2a+4.$
9. $\frac{27+x^3}{3+x}=9-3x+x^2.$

10. $\dfrac{x^6 - 16x^2}{x^3 + 4x} = x^3 - 4x.$

11. $\dfrac{x^3 - 64}{x - 4} = x^2 + 4x + 16.$

12. $\dfrac{1 - 8m^3}{1 - 2m} = 1 + 2m + 4m^2.$

13. $\dfrac{a^3 + 343}{a + 7} = a^2 - 7a + 49.$

14. $\dfrac{x^6 - y^6}{x^2 - y^2} = x^4 + x^2y^2 + y^4.$

15. $\dfrac{27 + x^3y^6}{3 + xy^2} = 9 - 3xy^2 + x^2y^4.$

16. $\dfrac{49a^2 - 121b^4}{7a - 11b^2} = 7a + 11b^2.$

17. $\dfrac{64m^3 + n^6}{4m + n^2} = 16m^2 - 4mn^2 + n^4$

18. $\dfrac{x^3 + 125y^3}{x + 5y} = x^2 - 5xy + 25y^2.$

19. $\dfrac{27x^3y^3 - 64z^3}{3xy - 4z} = 9x^2y^2 + 12xyz + 16z^2.$

20. $\dfrac{25a^4 - 81b^2c^6}{5a^2 - 9bc^3} = 5a^2 + 9bc^3.$

21. $\dfrac{343 + 125x^3y^3}{7 + 5xy} = 49 - 35xy + 25x^2y^2.$

22. $\dfrac{64m^3 - 216n^6}{4m - 6n^2} = 16m^2 + 24mn^2 + 36n^4.$

23. $\dfrac{729x^9y^3 + 512z^6}{9x^3y + 8z^2} = 81x^6y^2 - 72x^3yz^2 + 64z^4.$

Art. 101. — Page 50.

3. $\dfrac{a^6 - b^6}{a - b} = a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5.$

4. $\dfrac{x^6 - y^6}{x + y} = x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5.$

5. $\dfrac{m^7 + n^7}{m + n} = m^6 - m^5n + m^4n^2 - m^3n^3 + m^2n^4 - mn^5 + n^6.$

6. $\dfrac{m^7 - n^7}{m - n} = m^6 + m^5n + m^4n^2 + m^3n^3 + m^2n^4 + mn^5 + n^6.$

7. $\dfrac{1 - x^4}{1 - x} = 1 + x + x^2 + x^3.$

8. $\dfrac{x^4 - 16}{x - 2} = x^3 + 2x^2 + 4x + 8.$

9. $\dfrac{1 - a^5}{1 - a} = 1 + a + a^2 + a^3 + a^4.$

10. $\dfrac{a^5 + 1}{a + 1} = a^4 - a^3 + a^2 - a + 1.$

11. $\dfrac{1 - n^6}{1 - n} = 1 + n + n^2 + n^3 + n^4 + n^5.$

12. $\dfrac{x^4 - 81}{x - 3} = x^3 + 3x^2 + 9x + 27.$

13. $\dfrac{x^5-32}{x-2}=x^4+2x^3+4x^2+8x+16.$

14. $\dfrac{a^6-64}{a+2}=a^5-2a^4+4a^3-8a^2+16a-32.$

15. $\dfrac{a^{10}+b^{10}}{a^2+b^2}=a^8-a^6b^2+a^4b^4-a^2b^6+b^8.$

16. $\dfrac{x^7-128}{x-2}=x^6+2x^5+4x^4+8x^3+16x^2+32x+64.$

17. $\dfrac{x^5+243}{x+3}=x^4-3x^3+9x^2-27x+81.$

18. $\dfrac{m^4-16n^8}{m-2n^2}=m^3+2m^2n^2+4mn^4+8n^6.$

19. $\dfrac{x^8-y^8z^8}{x-yz}=x^7+x^6yz+x^5y^2z^2+x^4y^3z^3+x^3y^4z^4+x^2y^5z^5+xy^6z^6+y^7z^7.$

20. $\dfrac{32a^5+b^5}{2a+b}=16a^4-8a^3b+4a^2b^2-2ab^3+b^4.$

21. $\dfrac{m^5-243n^5}{m-3n}=m^4+3m^3n+9m^2n^2+27mn^3+81n^4.$

22. $\dfrac{256x^4-y^8}{4x+y^2}=64x^3-16x^2y^2+4xy^4-y^6.$

CHAPTER VIII.

Art. 104.—Pages 51, 52.

3. $x(x+5)$.
4. $3m^2(m-4)$.
5. $4a(4a^3-3)$.
6. $9c^3d(3cd+1)$.
7. $12m^2(5n^4-m)$.
8. $5x(x^2+2x+3)$.
9. $a^2(a^3-2a^2+3a-1)$.
10. $12x^2y(3x-5y^3-7x^2y)$.
11. $7mn(3m^2+5n^2-2)$.
12. $14x^2y^3(6-10xy+5x^2y^2)$.
13. $18a^2(3a^2b^3-4ac^2-5d)$.
14. $24c^3d^4(4c^2d+5d^3-6c^3)$.

Art. 105.—Page 52.

3. $b(a+x)+y(a+x)=(a+x)(b+y)$.
4. $c(a-m)+d(a-m)=(a-m)(c+d)$.
5. $x(x+2)-y(x+2)=(x+2)(x-y)$.
6. $x(x-a)-b(x-a)=(x-a)(x-b)$.
7. $a^2(a-b)+b^2(a-b)=(a-b)(a^2+b^2)$.
8. $a^2(a-b)-b^2(a-b)=(a-b)(a^2-b^2)$.
9. $x(x+a)-b(x+a)=(x+a)(x-b)$.
10. $m(x^2-y^2)+n(x^2-y^2)=(m+n)(x^2-y^2)$.
11. $x^2(x+1)+(x+1)=(x+1)(x^2+1)$.
12. $2x^2(3x+2)-3(3x+2)=(3x+2)(2x^2-3)$.
13. $4c(2x-3y)+d(2x-3y)=(4c+d)(2x-3y)$.
14. $3n(2-7m^2)-4m(2-7m^2)=(3n-4m)(2-7m^2)$.

Art. 110.—Page 54.

3. $(x+y)^2$.
4. $(2+m)^2$.
5. $(x-7)^2$.
6. $(a-5)^2$.
7. $(y+1)^2$.
8. $(m-1)^2$.
9. $(x^2+6)^2$.
10. $(n^3-10)^2$.
11. $(xy+8)^2$.

12. $(1-5ab)^2$.
13. $(4m-a)^2$.
14. $(a^2+a)^2$.
15. $(x^3-2x)^2$.
16. $(6m-3n)^2$.
17. $(2a+11b)^2$.
18. $(x^3+4x^2)^2$.
19. $(ab^3+9c)^2$.
20. $(5x-7yz)^2$.
21. $(3x^4-11x^2)^2$.
22. $(3a^2+10bc^2d)^2$.
23. $(8x^4-10x^3)^2$.
24. $(2a^2b+13ab^2)^2$.
25. $(4x^2-15mn)^2$.
26. $(a-b+1)^2$.
27. $(x+y-8)^2$.
28. $(x^2-x+3)^2$.

Art. 111.—Pages 55–57.

3. $(x+y)(x-y)$.
4. $(x+1)(x-1)$.
5. $(2+a)(2-a)$.
6. $(3m+2)(3m-2)$.
7. $(3x+4y)(3x-4y)$.
8. $(5a+b^2)(5a-b^2)$.
9. $(1+7xy)(1-7xy)$.
10. $(ab^3+c^2d^4)(ab^3-c^2d^4)$.
11. $(7m+10n^3)(7m-10n^3)$.
12. $(6x^2+9y)(6x^2-9y)$.
13. $(8a+11bc)(8a-11bc)$.
14. $(12xy^2+15z^3)(12xy^2-15z^3)$.
15. $[a+b+(c+d)][a+b-(c+d)]=(a+b+c+d)(a+b-c-d)$.
16. $(a-c+b)(a-c-b)$.
17. $[m+(x-y)][m-(x-y)]=(m+x-y)(m-x+y)$.
18. $[m^2+(m-1)][m^2-(m-1)]=(m^2+m-1)(m^2-m+1)$.
19. $[x-c+(y-d)][x-c-(y-d)]=(x-c+y-d)(x-c-y+d)$
20. $(a-3+b+2)(a-3-b-2)=(a+b-1)(a-b-5)$.
21. $(2x+m+x-m)(2x+m-x+m)=3x(x+2m)$.
22. $(3a+5+2a-3)(3a+5-2a+3)=(5a+2)(a+8)$.
26. $(x+y)^2-4=(x+y+2)(x+y-2)$.
27. $(a-b)^2-c^2=(a-b+c)(a-b-c)$.
28. $a^2-(b-c)^2=(a+b-c)(a-b+c)$.
29. $a^2-(b+c)^2=(a+b+c)(a-b-c)$.
30. $(c+d)^2-1=(c+d+1)(c+d-1)$.
31. $9-(x-y)^2=(3+x-y)(3-x+y)$.
32. $4m^4-(1-2b)^2=(2m^2+1-2b)(2m^2-1+2b)$.
33. $(2a-b)^2-9d^2=(2a-b+3d)(2a-b-3d)$.
34. $(a-m)^2-(b+n)^2=(a-m+b+n)(a-m-b-n)$.

35. $(x-c)^2-(y-d)^2=(x-c+y-d)(x-c-y+d)$.

36. $(a+m)^2-(b-n)^2=(a+m+b-n)(a+m-b+n)$.

37. $(a+c)^2-(b+d)^2=(a+c+b+d)(a+c-b-d)$.

Art. 114. — Pages 59, 60.

5. $(x+2)(x+3)$.
6. $(x-1)(x-2)$.
7. $(y+4)(y-2)$.
8. $(m-10)(m+3)$.
9. $(a-2)(a-9)$.
10. $(x+3)(x-2)$.
11. $(c+1)(c+8)$.
12. $(y-7)(y+5)$.
13. $(a-3)(a+16)$.
14. $(x-3)(x-7)$.
15. $(x+4)(x+9)$.
16. $(n-10)(n+9)$.
17. $(x-8)(x+2)$.
18. $(m+9)(m+7)$.
19. $(a-4)(a-11)$.
20. $(y+12)(y-5)$.
21. $(x-1)(x-10)$.
22. $(m+10)(m-8)$.
23. $(n+6)(n+17)$.
24. $(x-15)(x+6)$.
25. $(a-13)(a+2)$.
26. $(x+7)(x-6)$.
27. $(c-2)(c-16)$.
28. $(m-11)(m+3)$.
29. $(x+5)(x+15)$.
30. $(x+12)(x-8)$.
31. $(y+5)(y-22)$.
32. $(x-6)(x-13)$.
33. $(x+14)(x-7)$.
34. $(a+7)(a+15)$.
35. $(x-10)(x-13)$.
36. $(a^2+18)(a^2-8)$.
37. $(x^2+5)(x^2-24)$.
38. $(c^3+1)(c^3+11)$.
39. $(xy^3+12)(xy^3-10)$.
40. $(ab^2+9)(ab^2-16)$.
41. $(nx+5)(nx+20)$.
42. $(y^4-7)(y^4-13)$.
43. $(a^2b^2-8)(a^2b^2+6)$.
44. $(m^2+29)(m^2-3)$.
47. $(a-x)(a-2x)$.
48. $(x+11y)(x-6y)$.
49. $(1+6a)(1+7a)$.
50. $(m-8n)(m-7n)$.
51. $(a-8b)(a+7b)$.
52. $(ab+9c)(ab-5c)$.
53. $(1-5x)(1+2x)$.
54. $(a^2+11a)(a^2+4a)$.
55. $(z-13xy^2)(z+3xy^2)$.
56. $(a+b+1)(a+b+4)$.

57. $(1-a)(1-8a)$.

58. $(b^2+13a)(b^2-4a)$.

59. $(m-n+2)(m-n-1)$.

60. $(x^3-10x)(x^3+5x)$.

61. $(a+2b)(a+6b)$.

62. $(1-8xy)(1-5xy)$.

63. $(a-b-4)(a-b+1)$.

64. $(x^2y^2+12z)(x^2y^2-4z)$.

Art. 117.—Page 62.

3. $(a+x)(a^2-ax+x^2)$.

4. $(m-n)(m^2+mn+n^2)$.

5. $(x-1)(x^2+x+1)$.

6. $(ab+c)(a^2b^2-abc+c^2)$.

7. $(1-2x)(1+2x+4x^2)$.

8. $(a^2+b^2)(a^4-a^2b^2+b^4)$.

9. $(x^2+1)(x^4-x^2+1)$.

10. $(3x-1)(9x^2+3x+1)$.

11. $(2c^2-d^3)(4c^4+2c^2d^3+d^6)$.

12. $(3+2a)(9-6a+4a^2)$.

13. $(m-4n^2)(m^2+4mn^2+16n^4)$.

14. $(4x-5)(16x^2+20x+25)$.

15. $(5a+3m)(25a^2-15am+9m^2)$.

16. $(4cd^3+3)(16c^2d^6-12cd^3+9)$.

17. $(5-2ab^2)(25+10ab^2+4a^2b^4)$.

Art. 118.—Page 62.

2. $(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$.

3. $(x+1)(x^4-x^3+x^2-x+1)$.

4. $(1-a)(1+a+a^2+a^3+a^4)$.

5. $(m+n)(m^6-m^5n+m^4n^2-m^3n^3+m^2n^4-mn^5+n^6)$.

6. $(x-y)(x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5+y^6)$.

7. $(a-1)(a^6+a^5+a^4+a^3+a^2+a+1)$.

8. $(c-mn)(c^4+c^3mn+c^2m^2n^2+cm^3n^3+m^4n^4)$.

9. $(1+2n)(1-2n+4n^2-8n^3+16n^4)$.

10. $(3x-y)(81x^4+27x^3y+9x^2y^2+3xy^3+y^4)$.

11. $(x+2)(x^6-2x^5+4x^4-8x^3+16x^2-32x+64)$.

12. $(2-3a)(16+24a+36a^2+54a^3+81a^4)$.

Art. 120.—Pages 63, 64.

1. $6a^2x(x^2-a^2)=6a^2x(x+a)(x-a)$.

2. $(1-2x)^2$.

3. $(x^3+1)(x^3-1)=(x+1)(x^2-x+1)(x-1)(x^2+x+1)$.

4. $(a+3)(a+6)$.

5. $x(x+a)+b(x+a)=(x+a)(x+b)$.

6. $(m-8)(m+1)$.

7. $x(2x^4+1)$.

8. $5(a^3-1)=5(a-1)(a^2+a+1)$.

9. $(ab-cd)(a^4b^4+a^3b^3cd+a^2b^2c^2d^2+abc^3d^3+c^4d^4)$.

10. $(x^2+4)(x^2-4)=(x^2+4)(x+2)(x-2)$.

11. $a^2(a^3-a^2+a-1)=a^2[a^2(a-1)+(a-1)]=a^2(a-1)(a^2+1)$.

12. $3(x^2+9x+14)=3(x+2)(x+7)$.

13. $(x+2y-3z)(x-2y+3z)$.

14. $(a+10b)^2$.

15. $5abc(a-2b-3c)$.

16. $3a^2(a^2-7a+10)=3a^2(a-2)(a-5)$.

17. $(x+2yz)(x^2-2xyz+4y^2z^2)$.

18. $2a(a^4-1)=2a(a^2+1)(a^2-1)=2a(a^2+1)(a+1)(a-1)$.

19. $1-(a-b)^2=(1+a-b)(1-a+b)$.

20. $(x-1)(x-7)$.

21. $(1+3x)(1+9x)$.

22. $2xy(9x^2-y^2)=2xy(3x+y)(3x-y)$.

23. $x^2(x^7-1)=x^2(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$.

24. $(2xy^2+7)^2$.

25. $(a^3+10)(a^3-4)$.

26. $(a+2b)(a-20b)$.

27. $2xy(x^2+y^2+2xy-z^2)$
$$=2xy[(x+y)^2-z^2]=2xy(x+y+z)(x+y-z).$$

28. $6mn(2m^2-3mn+4n^2)$.

29. $4ab(8a^3+b^3)=4ab(2a+b)(4a^2-2ab+b^2)$

30. $(x^2+9)(x^2-9)=(x^2+9)(x+3)(x-3)$.

31. $y(1-y^8)=y(1+y^4)(1-y^4)$
$$=y(1+y^4)(1+y^2)(1-y^2)=y(1+y^4)(1+y^2)(1+y)(1-y).$$

32. $x^2(x+2)-(x+2)=(x+2)(x^2-1)=(x+2)(x+1)(x-1)$.

33. $x^2(1+7x-30x^2)=x^2(1+10x)(1-3x)$.

34. $(3x+y+x-2y)(3x+y-x+2y)=(4x-y)(2x+3y)$.

35. $(mx^3-13)(mx^3+5)$.

36. $5x^2(27x^3-1)=5x^2(3x-1)(9x^2+3x+1)$.

37. $2xy(x^2 - xy + 30y^2) = 2xy(x - 6y)(x + 5y)$.

38. $5x^2y\ (16y^4 - x^4) = 5x^2y\ (4y^2 + x^2)(4y^2 - x^2)$
$= 5x^2y\ (4y^2 + x^2)(2y + x)(2y - x)$.

39. $3ab\ (a^2 + 6a + 9) = 3ab\ (a + 3)^2$.

40. $(x^2+y^2-z^2+2xy)(x^2+y^2-z^2-2xy)=[(x+y)^2-z^2][(x-y)^2 - z^2]$
$= (x + y + z)(x + y - z)(x - y + z)(x - y - z)$.

41. $ac\ (ab - cd) - bd\ (ab - cd) = (ab - cd)(ac - bd)$.

42. $(a - 11b)(a - 3b)$.

43. $3xy\ (x^5 + y^5) = 3xy\ (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$.

44. $(2m^2 - 5n)^2$.

45. $3ab\ (a^2 + ab - 2b^2) = 3ab\ (a + 2b)(a - b)$.

46. $a^2(x^2 - y^2) - b^2(x^2 - y^2)$
$= (a^2 - b^2)(x^2 - y^2) = (a + b)(a - b)(x + y)(x - y)$.

47. $(a - 2b - 4)(a - 2b + 2)$.

48. $(10xy^2 + 9z)(10xy^2 - 9z)$.

49. $(a^3 + 8)(a^3 - 8) = (a + 2)(a^2 - 2a + 4)(a - 2)(a^2 + 2a + 4)$.

50. $(x^2 + x - 6)(x^2 - x + 6) = (x + 3)(x - 2)(x^2 - x + 6)$.

51. $(a^2 + 3a - 4)(a^2 + 3a - 10) = (a + 4)(a - 1)(a + 5)(a - 2)$.

52. $(4m + n + 2m - 3n)(4m + n - 2m + 3n)$
$= (6m - 2n)(2m + 4n) = 4(3m - n)(m + 2n)$.

53. $(a^2 - b^2 - c^2 + 2bc)(a^2 - b^2 - c^2 - 2bc)$
$= [a^2 - (b - c)^2][a^2 - (b + c)^2]$
$= (a + b - c)(a - b + c)(a + b + c)(a - b - c)$.

54. $(10 + 3m^2)(100 - 30m^2 + 9m^4)$.

55. $x^2(x - 1) - (x - 1) = (x^2 - 1)(x - 1) = (x + 1)(x - 1)(x - 1)$
$= (x + 1)(x - 1)^2$.

56. $3(a + b)(a - b) - (a - b)^2 = (a - b)[3(a + b) - (a - b)]$
$= (a - b)(3a + 3b - a + b) = (a - b)(2a + 4b) = 2(a - b)(a + 2b)$.

57. $(a - b)^2 - (c + d)^2 = (a - b + c + d)(a - b - c - d)$.

58. $(x^2 + 4 + 4x)(x^2 + 4 - 4x) = (x + 2)^2(x - 2)^2$.

59. $(x - y)(x^2 + xy + y^2) - 3xy(x - y) = (x - y)(x^2 + xy + y^2 - 3xy)$
$= (x - y)(x^2 - 2xy + y^2) = (x - y)(x - y)^2 = (x - y)^3$.

60. $(a^2 + a - 4 + 2)(a^2 + a - 4 - 2) = (a^2 + a - 2)(a^2 + a - 6)$
$= (a + 2)(a - 1)(a + 3)(a - 2)$.

CHAPTER IX.

Art. 126.—Page 66.

2. a^3x.
3. $3cd$.
4. $18a$.
5. $9mn$.
6. $14xyz^3$.
7. $15a^2$.
8. $12xy^2z^4$.
9. $24a^3b^4$.
10. $17a^2m^2$.

Art. 127.—Page 67.

3.

$$3ax^2 - 2a^2x = ax(3x - 2a)$$
$$a^2x^2 - 3abx = \underline{ax(ax - 3b)}$$
$$\text{H.C.F.} = ax.$$

4.

$$x^2 - y^2 = (x + y)(x - y)$$
$$x^3 + y^3 = \underline{(x+y)(x^2 - xy + y^2)}$$
$$\text{H.C.F.} = x + y.$$

5.

$$9a^4 - 4b^2 = (3a^2 + 2b)(3a^2 - 2b)$$
$$(3a^2 - 2b)^2 = \underline{(3a^2 - 2b)^2}$$
$$\text{H.C.F.} = 3a^2 - 2b.$$

6.

$$2x^6 - 2x^2 = 2x^2(x^2 - 1)(x^2 + 1)$$
$$6x^3 - 6x = \underline{6x\ (x^2 - 1)}$$
$$\text{H.C.F.} = 2x\ (x^2 - 1).$$

7.

$$3cx + 21c - 3dx - 21d = 3(x + 7)(c - d)$$
$$x^2 - 3x - 70 = \underline{(x + 7)(x - 10)}$$
$$\text{H.C.F.} = x + 7.$$

8.

$$m^3n + 2m^2n^2 + mn^3 = mn(m + n)^2$$
$$m^4n + mn^4 = \underline{mn(m + n)(m^2 - mn + n^2)}$$
$$\text{H.C.F.} = mn(m + n).$$

9.

$$3x^3 + 9x^2 - 120x = 3x(x + 8)(x - 5)$$
$$3ax^2 - 9ax - 30a = \underline{3a(x + 2)(x - 5)}$$
$$\text{H.C.F.} = 3(x - 5).$$

10.

$$\begin{aligned} 3xy-4y+3xz-4z &= (3x-4)(y+z) \\ 9x^2-16 &= \underline{(3x-4)(3x+4)} \\ \text{H. C. F.} &= 3x-4. \end{aligned}$$

11.

$$\begin{aligned} x^2-\quad x-42 &= (x+6)(x-7) \\ x^2-\ 4x-60 &= (x+6)(x-10) \\ x^2+12x+36 &= \underline{(x+6)^2} \\ \text{H. C. F.} &= x+6. \end{aligned}$$

12.

$$\begin{aligned} a^2-1 &= (a+1)(a-1) \\ a^3+1 &= (a+1)(a^2-a+1) \\ a^2+2a+1 &= \underline{(a+1)^2} \\ \text{H. C. F.} &= a+1. \end{aligned}$$

13.

$$\begin{aligned} 4x^2-12x+9 &= (2x-3)^2 \\ 4x^2-9 &= (2x-3)(2x+3) \\ 4m^2nx-6m^2n &= \underline{2m^2n(2x-3)} \\ \text{H. C. F.} &= 2x-3. \end{aligned}$$

14.

$$\begin{aligned} x^3-x &= x(x-1)(x+1) \\ x^3+9x^2-10x &= x(x-1)(x+10) \\ x^5-x &= \underline{x(x-1)(x^4+x^3+x^2+x+1)} \\ \text{H. C. F.} &= x(x-1). \end{aligned}$$

15.

$$\begin{aligned} a^3-8b^3 &= (a-2b)(a^2+2ab+4b^2) \\ a^2-ab-2b^2 &= (a-2b)(a+b) \\ a^2-4ab+4b^2 &= \underline{(a-2b)^2} \\ \text{H. C. F.} &= a-2b. \end{aligned}$$

16.

$$\begin{aligned} 2x^3+\ 2x^2-\ 4x &= 2x\,(x-1)(x+2) \\ 3x^4+\ 6x^3-\ 9x^2 &= 3x^2(x-1)(x+3) \\ 4x^5-20x^4+16x^3 &= \underline{4x^3(x-1)(x-4)} \\ \text{H. C. F.} &= x\,(x-1). \end{aligned}$$

17.

$$\begin{aligned} 8m^3 - 125 &= (2m-5)(4m^2+10m+25) \\ 4m^2 - \ \ 25 &= (2m-5)(2m+5) \\ 4m^2 - 20m + 25 &= \underline{(2m-5)^2} \\ \text{H.C.F.} &= 2m-5. \end{aligned}$$

18.

$$\begin{aligned} x^4 - 16 &= (x+2)(x-2)(x^2+4) \\ x^2 - x - 6 &= (x+2)(x-3) \\ (x^2-4)^2 &= \underline{(x+2)^2(x-2)^2} \\ \text{H.C.F.} &= x+2. \end{aligned}$$

19.

$$\begin{aligned} 3ax^6 - 3ax^5 &= 3ax^5(x-1) \\ ax^3 - 9ax^2 + 8ax &= ax(x-1)(x-8) \\ 2ax^5 - 2ax &= \underline{2ax(x-1)(x+1)(x^2+1)} \\ \text{H.C.F.} &= ax(x-1). \end{aligned}$$

20.

$$\begin{aligned} a^2 - b^2 &= (a-b)(a+b) \\ ab - b^2 + ac - bc &= (a-b)(b+c) \\ a^3 - a^2b + ab^2 - b^3 &= \underline{(a-b)(a^2+b^2)} \\ \text{H.C.F.} &= a-b. \end{aligned}$$

21.

$$\begin{aligned} 12ax - 3a + 8cx - 2c &= (4x-1)(3a+2c) \\ 16x^2 - 1 &= (4x-1)(4x+1) \\ 16x^2 - 8x + 1 &= \underline{(4x-1)^2} \\ \text{H.C.F.} &= 4x-1. \end{aligned}$$

Art. 131.—Pages 73, 74.

1.

$$\begin{array}{l} x^2 + x - 6)\,2x^2 - 11x + 14\,(2 \\ \qquad\qquad\ \ \underline{2x^2 + \ \ 2x - 12} \\ \qquad\qquad\qquad\ -13x + 26 \end{array}$$

Divide by -13,

$$\begin{array}{l} x - 2)\,x^2 + \ \ x - 6\,(x + 3 \\ \qquad\quad \underline{x^2 - 2x} \\ \qquad\qquad\quad 3x - 6 \\ \qquad\qquad\quad \underline{3x - 6} \end{array}$$

H. C. F. $= x - 2.$

2.

$$
\begin{array}{r}
6x^2-7x-24)\,12x^2+\ \ 8x-15\,(2 \\
\underline{12x^2-14x-48} \\
22x+33
\end{array}
$$

Divide by 11,

$$
\begin{array}{r}
2x+3)\,6x^2-\ \ 7x-24\,(3x-8 \\
\underline{6x^2+\ \ 9x\qquad} \\
-16x-24 \\
\underline{-16x-24}
\end{array}
$$

H. C. F. $= 2x+3$.

3.

$$
\begin{array}{r}
2a^2-5a+3)\,4a^3-\ \ 2a^2-\ \ 9a+\ \ 7\,(2a+4 \\
\underline{4a^3-10a^2+\ \ 6a\qquad} \\
8a^2-15a+\ \ 7 \\
\underline{8a^2-20a+12} \\
5a-\ \ 5
\end{array}
$$

Divide by 5,

$$
\begin{array}{r}
a-1)\,2a^2-5a+3\,(2a-3 \\
\underline{2a^2-2a\qquad} \\
-3a+3 \\
\underline{-3a+3}
\end{array}
$$

H. C. F. $= a-1$.

4.

$$
\begin{array}{r}
40x^2-\ \ 51ax+\ \ 14a^2 \\
\underline{3} \\
24x^2+11ax-28a^2)\,120x^2-153ax+\ \ 42a^2\,(5 \\
\underline{120x^2+\ \ 55ax-140a^2} \\
-208ax+182a^2
\end{array}
$$

Divide by $-26a$,

$$
\begin{array}{r}
8x-7a)\,24x^2+11ax-28a^2\,(3x+4a \\
\underline{24x^2-21ax\qquad\qquad} \\
32ax-28a^2 \\
\underline{32ax-28a^2}
\end{array}
$$

H. C. F. $= 8x-7a$.

5.

$$(6a^2b - 23ab + 20b) \div b = 6a^2 - 23a + 20.$$

$$(8a^3 - 22a^2 + 5a) \div a = 8a^2 - 22a + 5$$

$$\underline{\qquad\qquad\qquad\qquad 3}$$

$$6a^2 - 23a + 20)24a^2 - 66a + 15(4$$
$$\underline{24a^2 - 92a + 80}$$
$$26a - 65$$

Divide by 13,
$$2a - 5)6a^2 - 23a + 20(3a - 4$$
$$\underline{6a^2 - 15a}$$
$$-8a + 20$$
$$\underline{-8a + 20}$$

H.C.F. $= 2a - 5$.

6.

$$(x^3 - 5mx^2 + 4m^2x) \div x = x^2 - 5mx + 4m^2.$$

$$(x^4 - mx^3 + 3m^2x^2 - 3m^3x) \div x = x^3 - mx^2 + 3m^2x - 3m^3.$$

$$x^2 - 5mx + 4m^2)x^3 - mx^2 + 3m^2x - 3m^3(x + 4m$$
$$\underline{x^3 - 5mx^2 + 4m^2x}$$
$$4mx^2 - m^2x - 3m^3$$
$$\underline{4mx^2 - 20m^2x + 16m^3}$$
$$19m^2x - 19m^3$$

Divide by $19m^2$,
$$x - m)x^2 - 5mx + 4m^2(x - 4m$$
$$\underline{x^2 - mx}$$
$$-4mx + 4m^2$$
$$\underline{-4mx + 4m^2}$$

H.C.F. $= x(x - m)$.

7.

$$(5m^2n^2 + 58mn^2 + 33n^2) \div n^2 = 5m^2 + 58m + 33.$$

$$5m^2 + 58m + 33)10m^3 + 31m^2 - 20m - 21(2m - 17$$
$$\underline{10m^3 + 116m^2 + 66m}$$
$$-85m^2 - 86m - 21$$
$$\underline{-85m^2 - 986m - 561}$$
$$900m + 540$$

Divide by 180,
$$5m + 3)5m^2 + 58m + 33(m + 11$$
$$\underline{5m^2 + 3m}$$
$$55m + 33$$
$$\underline{55m + 33}$$

H.C.F. $= 5m + 3$.

8.

$$(2a^4 + 3a^3x - 9a^2x^2) \div a^2 = 2a^2 + 3ax - 9x^2.$$

$$\begin{array}{l}
2a^2 + 3ax - 9x^2)6a^3 - 17a^2x + 14ax^2 - 3x^3(3a - 13x \\
\qquad\qquad\qquad\quad 6a^3 + 9a^2x - 27ax^2 \\
\hline
\qquad\qquad\qquad\qquad -26a^2x + 41ax^2 - 3x^3 \\
\qquad\qquad\qquad\qquad -26a^2x - 39ax^2 + 117x^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad 80ax^2 - 120x^3
\end{array}$$

Divide by $40x^2$,

$$\begin{array}{l}
2a - 3x)2a^2 + 3ax - 9x^2(a + 3x \\
\qquad\quad 2a^2 - 3ax \\
\hline
\qquad\qquad\quad 6ax - 9x^2 \\
\qquad\qquad\quad 6ax - 9x^2 \\
\hline
\end{array}$$

H. C. F. $= 2a - 3x$.

9.

$$\begin{array}{l}
x^3 - 8)x^3 - 6x^2 + 11x - 6(1 \\
\qquad\quad x^3 \qquad\qquad\quad - 8 \\
\hline
\qquad\quad -6x^2 + 11x + 2
\end{array}$$

Changing signs,

$$\begin{array}{l}
\qquad\qquad\qquad x^3 \qquad\qquad\quad - 8 \\
\qquad\qquad\qquad\qquad\qquad\qquad\;\; 6 \\
\hline
6x^2 - 11x - 2)6x^3 \qquad\qquad - 48(x \\
\qquad\qquad\quad 6x^3 - 11x^2 - 2x \\
\hline
\qquad\qquad\qquad 11x^2 + 2x - 48 \\
\qquad\qquad\qquad\qquad\qquad\quad 6 \\
\hline
\qquad\qquad\qquad 66x^2 + 12x - 288(11 \\
\qquad\qquad\qquad 66x^2 - 121x - 22 \\
\hline
\qquad\qquad\qquad\qquad 133x - 266
\end{array}$$

Divide by 133,

$$\begin{array}{l}
x - 2)6x^2 - 11x - 2(6x + 1 \\
\qquad\quad 6x^2 - 12x \\
\hline
\qquad\qquad\qquad x - 2 \\
\qquad\qquad\qquad x - 2 \\
\hline
\end{array}$$

H. C. F. $= x - 2$.

10.

$$2x^3-3x^2-x+1)\,6x^3-x^2+3x-2\,(3$$
$$\underline{6x^3-9x^2-3x+3}$$
$$8x^2+6x-5$$

$$2x^3-3x^2-x+1$$
$$\underline{\qquad\qquad 4}$$
$$8x^2+6x-5)\,8x^3-12x^2-4x+4\,(x$$
$$\underline{8x^3+6x^2-5x}$$
$$-18x^2+x+4$$
$$\underline{\qquad\qquad 4}$$
$$-72x^2+4x+16\,(-9$$
$$\underline{-72x^2-54x+45}$$
$$58x-29$$

Divide by 29,

$$2x-1)\,8x^2+6x-5\,(4x+5$$
$$\underline{8x^2-4x}$$
$$10x-5$$
$$\underline{10x-5}$$

H. C. F. $=2x-1$.

11.

$$(6m^4-29m^3n+43m^2n^2-20mn^3)\div m=6m^3-29m^2n+43mn^2-20n^3.$$

$$6m^3-29m^2n+43mn^2-20n^3$$
$$\underline{\qquad\qquad 4}$$
$$8m^2-22mn+5n^2)\,24m^3-116m^2n+172mn^2-80n^3\,(3m$$
$$\underline{24m^3-66m^2n+15mn^2}$$
$$-50m^2n+157mn^2-80n^3$$
$$\underline{\qquad\qquad 4}$$
$$-200m^2n+628mn^2-320n^3\,(-25n$$
$$\underline{-200m^2n+550mn^2-125n^3}$$
$$78mn^2-195n^3$$

Divide by $39n^2$,

$$2m-5n)\,8m^2-22mn+5n^2\,(4m-n$$
$$\underline{8m^2-20mn}$$
$$-2mn+5n^2$$
$$\underline{-2mn+5n^2}$$

H. C. F. $=2m-5n$.

12.

$(ax^3+2ax^2+ax+2a)\div a = x^3+2x^2+x+2.$

$(3x^5-12x^3-3x^2-6x)\div 3x = x^4-4x^2-x-2.$

$$\begin{array}{r} x^3+2x^2+x+2)x^4 \qquad -4x^2-x-2(x-2 \\ x^4+2x^3+x^2+2x \\ \hline -2x^3-5x^2-3x-2 \\ -2x^3-4x^2-2x-4 \\ \hline -x^2-x+2 \end{array}$$

Changing signs, $\begin{array}{r} x^2+x-2)x^3+2x^2+x+2(x+1 \\ x^3+x^2-2x \\ \hline x^2+3x+2 \\ x^2+x-2 \\ \hline 2x+4 \end{array}$

Dividing by 2, $\begin{array}{r} x+2)x^2+x-2(x-1 \\ x^2+2x \\ \hline -x-2 \\ -x-2 \\ \hline \end{array}$

$\therefore$ H.C.F. $= x+2.$

13.

$(ax^4-ax^3-2ax^2+2ax)\div ax = x^3-x^2-2x+2.$

$(ax^5-3ax^4+2ax^3+ax^2-ax)\div ax = x^4-3x^3+2x^2+x-1.$

$$\begin{array}{r} x^3-x^2-2x+2)x^4-3x^3+2x^2+x-1(x-2 \\ x^4-x^3-2x^2+2x \\ \hline -2x^3+4x^2-x-1 \\ -2x^3+2x^2+4x-4 \\ \hline 2x^2-5x+3 \end{array}$$

$$\begin{array}{r} x^3-x^2-2x+2 \\ 2 \\ \hline 2x^2-5x+3)2x^3-2x^2-4x+4(x \\ 2x^3-5x^2+3x \\ \hline 3x^2-7x+4 \\ 2 \\ \hline 6x^2-14x+8(3 \\ 6x^2-15x+9 \\ \hline x-1 \end{array}$$

$$\begin{array}{r} x-1)2x^2-5x+3(2x-3 \\ 2x^2-2x \\ \hline -3x+3 \\ -3x+3 \\ \hline \end{array}$$

$\therefore$ H.C.F. $= ax(x-1)$
$= ax^2-ax.$

14.

$(2x^4-2x^3+4x^2+2x+6)\div 2 = x^4-x^3+2x^2+x+3.$

$(3x^4+6x^3-3x-6)\div 3 = x^4+2x^3-x-2.$

$$
\begin{array}{r|l}
x^4-x^3+2x^2+x+3) & x^4+2x^3 \qquad\qquad -x-2\,(1 \\
 & \underline{x^4-x^3+2x^2+x+3} \\
 & \qquad 3x^3-2x^2-2x-5 \\
 & \\
 & x^4-x^3+2x^2+x+3 \\
 & \underline{\qquad\qquad\qquad\qquad 3} \\
3x^3-2x^2-2x-5) & 3x^4-3x^3+6x^2+3x+9\,(x \\
 & \underline{3x^4-2x^3-2x^2-5x} \\
 & \quad -x^3+8x^2+8x+9 \\
 & \underline{\qquad\qquad\qquad\qquad 3} \\
 & \quad -3x^3+24x^2+24x+27\,(-1 \\
 & \quad \underline{-3x^3+2x^2+2x+5} \\
 & \qquad\qquad 22x^2+22x+22
\end{array}
$$

Dividing by 22,

$$
\begin{array}{r|l}
x^2+x+1) & 3x^3-2x^2-2x-5\,(3x-5 \\
 & \underline{3x^3+3x^2+3x} \\
 & \quad -5x^2-5x-5 \\
 & \quad \underline{-5x^2-5x-5}
\end{array}
$$

$\therefore$ H.C.F. $= x^2+x+1.$

15.

$$
\begin{array}{r|l}
a^4+a^3-6a^2+a+3) & a^4+2a^3-6a^2-a+2\,(1 \\
 & \underline{a^4+a^3-6a^2+a+3} \\
 & \qquad a^3 \qquad -2a-1 \\
 & \\
a^3-2a-1) & a^4+a^3-6a^2+a+3\,(a+1 \\
 & \underline{a^4 \qquad -2a^2-a} \\
 & \qquad a^3-4a^2+2a+3 \\
 & \qquad \underline{a^3 \qquad\quad -2a-1} \\
 & \qquad\quad -4a^2+4a+4
\end{array}
$$

Dividing by -4,

$$
\begin{array}{r|l}
a^2-a-1) & a^3 \qquad -2a-1\,(a+1 \\
 & \underline{a^3-a^2-a} \\
 & \qquad a^2-a-1 \\
 & \qquad \underline{a^2-a-1}
\end{array}
$$

$\therefore$ H.C.F. $= a^2-a-1.$

16.

$(x^5 - x^4 - 5x^3 + 2x^2 + 6x) \div x = x^4 - x^3 - 5x^2 + 2x + 6.$

$(x^5 + x^4 - x^3 - 2x^2 - 2x) \div x = x^4 + x^3 - x^2 - 2x - 2.$

$$\begin{array}{r}
x^4 - x^3 - 5x^2 + 2x + 6)x^4 + x^3 - x^2 - 2x - 2(1 \\
\underline{x^4 - x^3 - 5x^2 + 2x + 6} \\
2x^3 + 4x^2 - 4x - 8
\end{array}$$

Dividing by 2,

$$\begin{array}{r}
x^3 + 2x^2 - 2x - 4)x^4 - x^3 - 5x^2 + 2x + 6(x - 3 \\
\underline{x^4 + 2x^3 - 2x^2 - 4x} \\
-3x^3 - 3x^2 + 6x + 6 \\
\underline{-3x^3 - 6x^2 + 6x + 12} \\
3x^2 - 6
\end{array}$$

Dividing by 3,

$$\begin{array}{r}
x^2 - 2)x^3 + 2x^2 - 2x - 4(x + 2 \\
\underline{x^3 - 2x} \\
2x^2 - 4 \\
\underline{2x^2 - 4}
\end{array}$$

$\therefore$ H. C. F. $= x(x^2 - 2)$
$= x^3 - 2x.$

17.

$(15a^2x^3 - 20a^2x^2 - 65a^2x - 30a^2) \div 5a^2 = 3x^3 - 4x^2 - 13x - 6.$

$(12bx^3 + 20bx^2 - 16bx - 16b) \div 4b = 3x^3 + 5x^2 - 4x - 4.$

$$\begin{array}{r}
3x^3 - 4x^2 - 13x - 6)3x^3 + 5x^2 - 4x - 4(1 \\
\underline{3x^3 - 4x^2 - 13x - 6} \\
9x^2 + 9x + 2 \\
\\
3x^3 - 4x^2 - 13x - 6 \\
\underline{3} \\
9x^2 + 9x + 2)9x^3 - 12x^2 - 39x - 18(x \\
\underline{9x^3 + 9x^2 + 2x} \\
-21x^2 - 41x - 18 \\
\underline{3} \\
-63x^2 - 123x - 54(-7 \\
\underline{-63x^2 - 63x - 14} \\
-60x - 40
\end{array}$$

Dividing by -20,

$$\begin{array}{r}
3x + 2)9x^2 + 9x + 2(3x + 1 \\
\underline{9x^2 + 6x} \\
3x + 2 \\
\underline{3x + 2}
\end{array}$$

$\therefore$ H. C. F. $= 3x + 2.$

18.

$$\begin{array}{r}
a^4 + a^3x + a^2x^2 + ax^3 - 4x^4)\,a^4 + 2a^3x + 3a^2x^2 + 4ax^3 - 10x^4\,(1 \\
\underline{a^4 + a^3x + a^2x^2 + ax^3 - 4x^4} \\
a^3x + 2a^2x^2 + 3ax^3 - 6x^4
\end{array}$$

Dividing by x,

$$\begin{array}{r}
a^3 + 2a^2x + 3ax^2 - 6x^3)\,a^4 + a^3x + a^2x^2 + ax^3 - 4x^4\,(a - x \\
\underline{a^4 + 2a^3x + 3a^2x^2 - 6ax^3} \\
-a^3x - 2a^2x^2 + 7ax^3 - 4x^4 \\
\underline{-a^3x - 2a^2x^2 - 3ax^3 + 6x^4} \\
10ax^3 - 10x^4
\end{array}$$

Dividing by $10x^3$,

$$\begin{array}{r}
a - x)\,a^3 + 2a^2x + 3ax^2 - 6x^3\,(a^2 + 3ax \\
\underline{a^3 - a^2x} \quad + 6x^2 \\
3a^2x + 3ax^2 - 6x^3 \\
\underline{3a^2x - 3ax^2} \\
6ax^2 - 6x^3 \\
\underline{6ax^2 - 6x^3}
\end{array}$$

$\therefore$ H. C. F. $= a - x$.

19.

$(x^5 + 3x^4 + 2x) \div x = x^4 + 3x^3 + 2$.

$$\begin{array}{r}
x^4 + x^3 + x^2 - 1)\,x^4 + 3x^3 + 2\,(1 \\
\underline{x^4 + x^3 + x^2 - 1} \\
2x^3 - x^2 + 3
\end{array}$$

$$\begin{array}{r}
x^4 + x^3 + x^2 - 1 \\
\underline{2} \\
2x^3 - x^2 + 3)\,2x^4 + 2x^3 + 2x^2 - 2\,(x \\
\underline{2x^4 - x^3 + 3x } \\
3x^3 + 2x^2 - 3x - 2 \\
\underline{2} \\
6x^3 + 4x^2 - 6x - 4\,(3 \\
\underline{6x^3 - 3x^2 + 9} \\
7x^2 - 6x - 13
\end{array}$$

$$\begin{array}{r}
2x^3 - x^2 + 3 \\
\underline{7} \\
7x^2 - 6x - 13)\,14x^3 - 7x^2 + 21\,(2x \\
\underline{14x^3 - 12x^2 - 26x} \\
5x^2 + 26x + 21 \\
\underline{7} \\
35x^2 + 182x + 147\,(5 \\
\underline{35x^2 - 30x - 65} \\
212x + 212
\end{array}$$

Dividing by 212,

$$\begin{array}{r}
x + 1)\,7x^2 - 6x - 13\,(7x - 13 \\
\underline{7x^2 + 7x} \\
-13x - 13 \\
\underline{-13x - 13}
\end{array}$$

$\therefore$ H. C. F. $= x + 1$.

20.

$$\begin{array}{r}
x^4-x^3y-3x^2y^2+5xy^3-6y^4)3x^4-5x^3y-x^2y^2-7xy^3+10y^4(3\\
\underline{3x^4-3x^3y-9x^2y^2+15xy^3-18y^4}\\
-2x^3y+8x^2y^2-22xy^3+28y^4
\end{array}$$

Dividing by $-2y$,

$$\begin{array}{r}
x^3-4x^2y+11xy^2-14y^3)x^4-x^3y-3x^2y^2+5xy^3-6y^4(x+3y\\
\underline{x^4-4x^3y+11x^2y^2-14xy^3}\\
3x^3y-14x^2y^2+19xy^3-6y^4\\
\underline{3x^3y-12x^2y^2+33xy^3-42y^4}\\
-2x^2y^2-14xy^3+36y^4
\end{array}$$

Dividing by $-2y^2$,

$$\begin{array}{r}
x^2+7xy-18y^2)x^3-4x^2y+11xy^2-14y^3(x-11y\\
\underline{x^3+7x^2y-18xy^2}\\
-11x^2y+29xy^2-14y^3\\
\underline{-11x^2y-77xy^2+198y^3}\\
106xy^2-212y^3
\end{array}$$

Dividing by $106y^2$,

$$\begin{array}{r}
x-2y)x^2+7xy-18y^2(x+9y\\
\underline{x^2-2xy}\\
9xy-18y^2\\
\underline{9xy-18y^2}
\end{array}$$

$\therefore$ H. C. F. $= x-2y$.

21.

$$\begin{array}{r}
2x^4-5x^3+5x^2-5x+3)2x^4-7x^3+4x^2+5x-3(1\\
\underline{2x^4-5x^3+5x^2-5x+3}\\
-2x^3-x^2+10x-6
\end{array}$$

Changing signs,

$$\begin{array}{r}
2x^3+x^2-10x+6)2x^4-5x^3+5x^2-5x+3(x-3\\
\underline{2x^4+x^3-10x^2+6x}\\
-6x^3+15x^2-11x+3\\
\underline{-6x^3-3x^2+30x-18}\\
18x^2-41x+21
\end{array}$$

$$\begin{array}{r}
2x^3+x^2-10x+6\\
9\\
18x^2-41x+21)\overline{18x^3+9x^2-90x+54}(x\\
\underline{18x^3-41x^2+21x}\\
50x^2-111x+54\\
9\\
\overline{450x^2-999x+486}(25\\
\underline{450x^2-1025x+525}\\
26x-39
\end{array}$$

Dividing by 13,

$$\begin{array}{r}
2x-3)18x^2-41x+21(9x-7\\
\underline{18x^2-27x}\\
-14x+21\\
\underline{-14x+21}
\end{array}$$

$\therefore$ H. C. F. $= 2x-3$.

22.

$$
\begin{array}{r}
3a^4-2a^3b+2a^2b^2-5ab^3-2b^4)\,6a^4-\ a^3b+2a^2b^2-\ 2ab^3-\ b^4\,(2 \\
\underline{6a^4-4a^3b+4a^2b^2-10ab^3-4b^4} \\
3a^3b-2a^2b^2+\ 8ab^3+3b^4
\end{array}
$$

Dividing by b,

$$
\begin{array}{r}
3a^3-2a^2b+8ab^2+3b^3)\,3a^4-2a^3b+2a^2b^2-5ab^3-2b^4\,(a \\
\underline{3a^4-2a^3b+8a^2b^2+3ab^3} \\
-6a^2b^2-8ab^3-2b^4
\end{array}
$$

Dividing by $-2b^2$,

$$
\begin{array}{r}
3a^2+4ab+b^2)\,3a^3-2a^2b+\ 8ab^2+3b^3\,(a-2b \\
\underline{3a^3+4a^2b+\ \ ab^2} \\
-6a^2b+\ 7ab^2+3b^3 \\
\underline{-6a^2b-\ 8ab^2-2b^3} \\
15ab^2+5b^3
\end{array}
$$

Dividing by $5b^2$,

$$
\begin{array}{r}
3a+b)\,3a^2+4ab+b^2\,(a+b \\
\underline{3a^2+\ ab} \\
3ab+b^2 \\
\underline{3ab+b^2}
\end{array}
$$

$\therefore$ H. C. F. $=3a+b$.

Art. 132.—Page 74.

1.

$$
\begin{array}{r}
2x^2-5x-42)\,4x^2+\ 8x-21\,(2 \\
\underline{4x^2-10x-84} \\
18x+63
\end{array}
$$

Dividing by 9,

$$
\begin{array}{r}
2x+7)\,2x^2-\ 5x-42\,(x-6 \\
\underline{2x^2+\ 7x} \\
-12x-42 \\
\underline{-12x-42}
\end{array}
$$

$$
\begin{array}{r}
2x+7)\,6x^2+23x+7\,(3x+1 \\
\underline{6x^2+21x} \\
2x+7 \\
\underline{2x+7}
\end{array}
$$

$\therefore$ H. C. F. $=2x+7$.

2.

$$\begin{array}{r}
14x^2 - 39x + 10 \\
6 \\
\hline
12x^2 - 28x - 5)84x^2 - 234x + 60(7 \\
84x^2 - 196x - 35 \\
\hline
-38x + 95
\end{array}$$

Dividing by -19,

$$\begin{array}{r}
2x - 5)12x^2 - 28x - 5(6x + 1 \\
12x^2 - 30x \\
\hline
2x - 5 \\
2x - 5 \\
\hline
\end{array}$$

$$\begin{array}{r}
2x - 5)10x^2 - 11x - 35(5x + 7 \\
10x^2 - 25x \\
\hline
14x - 35 \\
14x - 35 \\
\hline
\end{array}$$

$\therefore$ H. C. F. $= 2x - 5$.

3.

$$\begin{array}{r}
15m^2 + 4mn - 4n^2 \\
2 \\
\hline
6m^2 + 7mn + 2n^2)30m^2 + 8mn - 8n^2(5 \\
30m^2 + 35mn + 10n^2 \\
\hline
-27mn - 18n^2
\end{array}$$

Dividing by $-9n$,

$$\begin{array}{r}
3m + 2n)6m^2 + 7mn + 2n^2(2m + n \\
6m^2 + 4mn \\
\hline
3mn + 2n^2 \\
3mn + 2n^2 \\
\hline
\end{array}$$

$$\begin{array}{r}
3m + 2n)3m^3 - 7m^2n - 12mn^2 - 4n^3(m^2 - 3mn - 2n^2 \\
3m^3 + 2m^2n \\
\hline
-9m^2n - 12mn^2 - 4n^3 \\
-9m^2n - 6mn^2 \\
\hline
-6mn^2 - 4n^3 \\
-6mn^2 - 4n^3 \\
\hline
\end{array}$$

$\therefore$ H. C. F. $= 3m + 2n$.

4.

$$\begin{array}{r}
6a^2+13a-5)\,6a^3+19a^2+\ \ 8a-5\,(a+1\\
\underline{6a^3+13a^2-\ \ 5a}\\
6a^2+13a-5\\
\underline{6a^2+13a-5}
\end{array}$$

$$\begin{array}{r}
3a^3+\ \ 2a^2+\ \ 2a-\ \ 1\\
\underline{2}\\
6a^2+13a-5)\,6a^3+\ \ 4a^2+\ \ 4a-\ \ 2\,(a\\
\underline{6a^3+13a^2-\ \ 5a}\\
-\ \ 9a^2+\ \ 9a-\ \ 2\\
\underline{2}\\
-18a^2+18a-\ \ 4\,(-3\\
\underline{-18a^2-39a+15}\\
57a-19
\end{array}$$

Dividing by 19,

$$\begin{array}{r}
3a-1)\,6a^2+13a-5\,(2a+5\\
\underline{6a^2-\ \ 2a}\\
15a-5\\
\underline{15a-5}
\end{array}$$

$\therefore$ H.C.F. $=3a-1.$

5.

$$\begin{array}{r}
x^3+3x^2-6x-8)\,x^3+5x^2+2x-8\,(1\\
\underline{x^3+3x^2-6x-8}\\
2x^2+8x
\end{array}$$

Dividing by $2x$,

$$\begin{array}{r}
x+4)\,x^3+3x^2-6x-8\,(x^2-x-2\\
\underline{x^3+4x^2}\\
-\ \ x^2-6x-8\\
\underline{-\ \ x^2-4x}\\
-2x-8\\
\underline{-2x-8}
\end{array}$$

$$\begin{array}{r}
x+4)\,x^3-3x^2-16x+48\,(x^2-7x+12\\
\underline{x^3+4x^2}\\
-7x^2-16x+48\\
\underline{-7x^2-28x}\\
12x+48\\
\underline{12x+48}
\end{array}$$

$\therefore$ H.C.F. $=x+4.$

6.

$$\begin{array}{r} x^3-7x+6)\,x^3+3x^2-16x+12\,(1 \\ \underline{x^3 \qquad\quad -7x+6} \\ 3x^2-9x+6 \end{array}$$

Dividing by 3,

$$\begin{array}{r} x^2-3x+2)\,x^3 \qquad -7x+6\,(x+3 \\ \underline{x^3-3x^2+2x} \\ 3x^2-9x+6 \\ \underline{3x^2-9x+6} \end{array}$$

$$\begin{array}{r} x^2-3x+2)\,x^3-5x^2+7x-3\,(x-2 \\ \underline{x^3-3x^2+2x} \\ -2x^2+5x-3 \\ \underline{-2x^2+6x-4} \\ -x+1 \end{array}$$

Changing signs,

$$\begin{array}{r} x-1)\,x^2-3x+2\,(x-2 \\ \underline{x^2-x} \\ -2x+2 \\ \underline{-2x+2} \end{array}$$

$\therefore$ H.C.F. $= x-1$.

7.

$$\begin{array}{r} 2a^3-3a^2-5a+6)\,2a^3+3a^2-8a-12\,(1 \\ \underline{2a^3-3a^2-5a+6} \\ 6a^2-3a-18 \end{array}$$

Dividing by 3,

$$\begin{array}{r} 2a^2-a-6)\,2a^3-3a^2-5a+6\,(a-1 \\ \underline{2a^3-a^2-6a} \\ -2a^2+a+6 \\ \underline{-2a^2+a+6} \end{array}$$

$$\begin{array}{r} 2a^2-a-6)\,2a^3-a^2-12a-9\,(a \\ \underline{2a^3-a^2-6a} \\ -6a-9 \end{array}$$

Dividing by -3,

$$\begin{array}{r} 2a+3)\,2a^2-a-6\,(a-2 \\ \underline{2a^2+3a} \\ -4a-6 \\ \underline{-4a-6} \end{array}$$

$\therefore$ H.C.F. $= 2a+3$.

CHAPTER X.

Art. 136.—Page 76.

2.

$$\begin{aligned} 6a^3b &= 2\cdot 3\cdot a^3b \\ a^2b^2 &= \underline{\phantom{2\cdot 3\cdot{}} a^2b^2} \\ \therefore \text{L. C. M.} &= 2\cdot 3\cdot a^3b^2 \\ &= 6a^3b^2. \end{aligned}$$

3.

$$\begin{aligned} 10x^2y &= 2\cdot 5\cdot x^2y \\ 12y^3z &= \underline{2^2\cdot 3\cdot y^3z} \\ \therefore \text{L. C. M.} &= 2^2\cdot 3\cdot 5\cdot x^2y^3z \\ &= 60x^2y^3z. \end{aligned}$$

4.

$$\begin{aligned} 30m^2 &= 2\cdot 3\cdot 5\cdot m^2 \\ 27n^2 &= \underline{3^3\cdot n^2} \\ \therefore \text{L. C. M.} &= 2\cdot 3^3\cdot 5\cdot m^2n^2 \\ &= 270m^2n^2. \end{aligned}$$

5.

$$\begin{aligned} 6ab &= 2\cdot 3\cdot ab \\ 10bc &= 2\cdot 5\cdot bc \\ 14ca &= \underline{2\cdot 7\cdot a\,c} \\ \therefore \text{L. C. M.} &= 2\cdot 3\cdot 5\cdot 7\cdot abc \\ &= 210abc. \end{aligned}$$

6.

$$\begin{aligned} a^5b^2 &= a^5b^2 \\ 9a^3b^4 &= 3^2\cdot a^3b^4 \\ 12a^2b^3 &= \underline{2^2\cdot 3\cdot a^2b^3} \\ \therefore \text{L. C. M.} &= 2^2\cdot 3^2\cdot a^5b^4 \\ &= 36a^5b^4. \end{aligned}$$

7.

$$\begin{aligned} 16x^2y &= 2^4\cdot x^2y \\ 42y^3z &= \underline{2\cdot 3\cdot 7\cdot y^3z} \\ \therefore \text{L. C. M.} &= 2^4\cdot 3\cdot 7\cdot x^2y^3z \\ &= 336x^2y^3z. \end{aligned}$$

8.

$$\begin{aligned} 8c^2d^3 &= 2^3\cdot c^2d^3 \\ 10ac &= 2\cdot 5\cdot a\,c \\ 18a^2d &= \underline{2\cdot 3^2\cdot a^2\,d} \\ \therefore \text{L. C. M.} &= 2^3\cdot 3^2\cdot 5\cdot a^2c^2d^3 \\ &= 360a^2c^2d^3. \end{aligned}$$

9.

$$\begin{aligned} 24m^3x^2 &= 2^3\cdot 3\cdot m^3\,x^2 \\ 30n^2y &= 2\cdot 3\cdot 5\cdot n^2\,y \\ 32xy^2 &= \underline{2^5\cdot x\,y^2} \\ \therefore \text{L. C. M.} &= 2^5\cdot 3\cdot 5\cdot m^3n^2x^2y^2 \\ &= 480m^3n^2x^2y^2. \end{aligned}$$

10.

$$\begin{aligned} 36xy^2z^3 &= 2^2\cdot 3^2\cdot x\,y^2z^3 \\ 63x^3yz^2 &= 3^2\cdot 7\cdot x^3y\,z^2 \\ 28x^2y^3z &= \underline{2^2\cdot 7\cdot x^2y^3z} \\ \therefore \text{L. C. M.} &= 2^2\cdot 3^2\cdot 7\cdot x^3y^3z^3 \\ &= 252x^3y^3z^3. \end{aligned}$$

11.

$$\begin{aligned} 40a^2bd^3 &= 2^3\cdot 5\cdot a^2b\,d^3 \\ 90ac^3d^4 &= 2\cdot 3^2\cdot 5\cdot a\,c^3d^4 \\ 54b^3cd^2 &= \underline{2\cdot 3^3\cdot b^3c\,d^2} \\ \therefore \text{L. C. M.} &= 2^3\cdot 3^3\cdot 5\cdot a^2b^3c^3d^4 \\ &= 1080a^2b^3c^3d^4. \end{aligned}$$

Art. 137.—Pages 76, 77.

2.

$$\begin{aligned} x^2 - y^2 &= (x+y)(x-y) \\ xy - y^2 &= y(x-y) \\ \therefore \text{L. C. M.} &= y(x+y)(x-y) \\ &= y(x^2-y^2). \end{aligned}$$

3.

$$\begin{aligned} x^2 - 1 &= (x+1)(x-1) \\ x^2-7x-8 &= (x+1)(x-8) \\ \therefore \text{L. C. M.} &= (x+1)(x-1)(x-8) \\ &= (x^2-1)(x-8). \end{aligned}$$

4.

$$\begin{aligned} 8a^2b + 8ab^2 &= 2^3 \cdot ab(a+b) \\ 6a - 6b &= 2 \cdot 3 \cdot (a-b) \\ \therefore \text{L. C. M.} &= 2^3 \cdot 3 \cdot ab(a+b)(a-b) \\ &= 24ab(a^2-b^2). \end{aligned}$$

5.

$$\begin{aligned} m^2 - n^2 &= (m+n)(m-r) \\ m^3 - n^3 &= (m-n)(m^2+mn+n^2) \\ \therefore \text{L. C. M.} &= (m+n)(m-n)(m^2+mn+n^2) \\ &= (m+n)(m^3-n^3). \end{aligned}$$

6.

$$\begin{aligned} a-b &= a-b \\ a^2-4ab+3b^2 &= (a-b)(a-3b) \\ \therefore \text{L. C. M.} &= (a-b)(a-3b) \\ &= a^2-4ab+3b^2. \end{aligned}$$

7.

$$\begin{aligned} x^2-2xy+y^2 &= (x-y)^2 \\ x^3y-xy^3 &= xy(x+y)(x-y) \\ \therefore \text{L. C. M.} &= xy(x+y)(x-y)^2. \end{aligned}$$

8.

$$\begin{aligned} 2a^2+2ab &= 2 \cdot a(a+b) \\ 3ab-3b^2 &= 3 \cdot b(a-b) \\ 4a^2c-4b^2c &= 2^2 \cdot c(a+b)(a-b) \\ \therefore \text{L. C. M.} &= 2^2 \cdot 3 \cdot abc(a+b)(a-b) \\ &= 12abc(a^2-b^2). \end{aligned}$$

9.

$$\begin{aligned} x^2+2ax-35a^2 &= (x-5a)(x+7a) \\ x^2-2ax-15a^2 &= (x+3a)(x-5a) \\ \therefore \text{L. C. M.} &= (x+3a)(x-5a)(x+7a). \end{aligned}$$

10.

$$\begin{aligned} mn + n^2 &= n\,(m+n) \\ mn - n^2 &= n\qquad\quad (m-n) \\ m^2 - n^2 &= \underline{\quad (m+n)(m-n)} \\ \therefore \text{L. C. M.} &= n\,(m+n)(m-n) \\ &= n\,(m^2-n^2). \end{aligned}$$

11.

$$\begin{aligned} ax - 2a + bx - 2b &= (a+b)\qquad\quad (x-2) \\ a^2 - 2ab - 3b^2 &= \underline{(a+b)(a-3b)\qquad\quad} \\ \therefore \text{L. C. M.} &= (a+b)(a-3b)(x-2). \end{aligned}$$

12.

$$\begin{aligned} ax^2 + a^2x &= ax\,(x+a) \\ x^2 - a^2 &= \quad (x+a)(x-a) \\ x^3 - a^3 &= \underline{\qquad\qquad (x-a)(x^2+ax+a^2)} \\ \therefore \text{L. C. M.} &= ax\,(x+a)(x-a)(x^2+ax+a^2) \\ &= ax\,(x+a)(x^3-a^3). \end{aligned}$$

13.

$$\begin{aligned} 8\,(a^2-b^2) &= 2^3 \cdot \quad (a+b)\ (a-b) \\ 6\,(a+b)^2 &= 2\ \cdot 3 \cdot (a+b)^2 \\ 12\,(a-b)^2 &= \underline{2^2 \cdot 3 \cdot \qquad\quad (a-b)^2} \\ \therefore \text{L. C. M.} &= 2^3 \cdot 3 \cdot (a+b)^2\,(a-b)^2 \\ &= 24\,(a+b)^2\,(a-b)^2. \end{aligned}$$

14.

$$\begin{aligned} x^3 - 10x^2 + 21x &= x\,(x-3)(x-7) \\ ax^2 + 5ax - 24a &= \underline{a\ \ (x-3)\qquad\quad (x+8)} \\ \therefore \text{L. C. M.} &= ax\,(x-3)(x-7)(x+8). \end{aligned}$$

15.

$$\begin{aligned} x^2 - 1 &= (x+1)\ (x-1) \\ x^2 - 2x + 1 &= \qquad\quad (x-1)^2 \\ x^2 + 2x + 1 &= \underline{(x+1)^2\qquad\qquad} \\ \therefore \text{L. C. M.} &= (x+1)^2\,(x-1)^2 \\ &= (x^2-1)^2 \\ &= x^4 - 2x^2 + 1. \end{aligned}$$

16.

$$\begin{aligned}
2-\ 2x^2 &= 2\cdot\quad(1+x)(1-x)\\
4-\ 4x\ &= 2^2\cdot\qquad\quad(1-x)\\
8+\ 8x\ &= 2^3\cdot\quad(1+x)\\
12+12x^2 &= 2^2\cdot 3\cdot\qquad\qquad\qquad(1+x^2)\\
\therefore \text{L. C. M.} &= 2^3\cdot 3\cdot(1+x)(1-x)(1+x^2)\\
&= 24(1-x^2)(1+x^2)\\
&= 24(1-x^4).
\end{aligned}$$

17.

$$\begin{aligned}
x^2+5x+\ 4 &= (x+1)\qquad\qquad\quad(x+4)\\
x^2+2x-\ 8 &= \qquad(x-2)\qquad(x+4)\\
x^2+7x+12 &= \qquad\qquad(x+3)(x+4)\\
\therefore \text{L. C. M.} &= (x+1)(x-2)(x+3)(x+4).
\end{aligned}$$

18.

$$\begin{aligned}
&a\qquad\qquad(x-b)(x-c)\\
&\ b\ (x-a)\qquad\ (x-c)\\
&\ c(x-a)(x-b)\\
\therefore \text{L. C. M.} &= abc(x-a)(x-b)(x-c).
\end{aligned}$$

19.

$$\begin{aligned}
(2m-1)^2 &= \qquad\qquad(2m-1)^2\\
4m^2-1 &= (2m+1)(2m-1)\\
8m^3-1 &= \qquad\qquad(2m-1)\ (4m^2+2m+1)\\
\therefore \text{L. C. M.} &= (2m+1)(2m-1)^2(4m^2+2m+1).
\end{aligned}$$

20.

$$\begin{aligned}
a^2+a\ &= a\ (a+1)\\
a^4-a^2 &= a^2(a+1)(a\ -1)\\
a^6+a^3 &= a^3(a+1)\qquad\quad(a^2-a+1)\\
\therefore \text{L. C. M.} &= a^3(a+1)(a\ -1)(a^2-a+1)\\
&= a^3(a-1)(a^3+1).
\end{aligned}$$

21.

$$\begin{aligned}
a^2-4a+\ 3 &= (a-1)(a-3)\\
a^2+\ a-12 &= \qquad\quad(a-3)(a+4)\\
a^2-\ a-20 &= \qquad\qquad\quad(a+4)(a-5)\\
\therefore \text{L. C. M.} &= (a-1)(a-3)(a+4)(a-5).
\end{aligned}$$

22.

$$\begin{aligned} 1 - x^4 &= (1+x)\,(1-x)\,(1+x^2) \\ 1 + 2x^2 + x^4 &= (1+x^2)^2 \\ 1 - 2x^2 + x^4 &= (1+x)^2(1-x)^2 \\ \therefore \text{L. C. M.} &= (1+x)^2(1-x)^2(1+x^2)^2. \end{aligned}$$

23.

$$\begin{aligned} (a+b)^2 - c^2 &= (a+b+c)(a+b-c) \\ (a-c)^2 - b^2 &= (a+b-c)(a-b-c) \\ \therefore \text{L. C. M.} &= (a+b+c)(a+b-c)(a-b-c). \end{aligned}$$

24.

$$\begin{aligned} ax - ay - bx + by &= (a-b)(x-y) \\ (x-y)^2 &= (x-y)^2 \\ 3a^2b - 3ab^2 &= 3ab(a-b) \\ \therefore \text{L. C. M.} &= 3ab(a-b)(x-y)^2. \end{aligned}$$

25.

$$\begin{aligned} 9x^3 + 12x^2 + 4x &= x\,(3x+2)^2 \\ 18ax^4 - 12ax^3 + 8ax^2 &= 2ax^2\,(9x^2 - 6x + 4) \\ 27x^3 + 8 &= (3x+2)\,(9x^2 - 6x + 4) \\ \therefore \text{L. C. M.} &= 2ax^2(3x+2)^2(9x^2 - 6x + 4). \end{aligned}$$

26.

$$\begin{aligned} x^2 - y^2 - z^2 + 2yz &= (x+y-z)(x-y+z) \\ x^2 - y^2 + z^2 + 2xz &= (x+y+z)(x-y+z) \\ \therefore \text{L. C. M.} &= (x+y+z)(x+y-z)(x-y+z). \end{aligned}$$

Art. 139.—Page 79.

2.

$$\begin{array}{r} 2x^2 + x - 6)\,4x^2 - 8x + 3\,(2 \\ \underline{4x^2 + 2x - 12} \\ -10x + 15 \end{array}$$

Dividing by -5,

$$\begin{array}{r} 2x - 3)\,2x^2 + x - 6\,(x+2 \\ \underline{2x^2 - 3x} \\ 4x - 6 \\ \underline{4x - 6} \end{array}$$

$\therefore$ H.C.F. $= 2x - 3$.

$$\begin{array}{r} 4x^2 - 8x + 3 \\ \underline{x + 2} \\ 4x^3 - 8x^2 + 3x \\ \underline{8x^2 - 16x + 6} \end{array}$$

$\therefore$ L. C. M. $= 4x^3 \quad - 13x + 6.$

3.

$$
\begin{array}{r}
6x^2+13x-28)\,12x^2-31x+20\,(2 \\
\underline{12x^2+26x-56} \\
-57x+76
\end{array}
$$

Dividing by -19,

$$
\begin{array}{r}
3x-4)\,6x^2+13x-28\,(2x+7 \\
\underline{6x^2-\ \ 8x} \\
21x-28 \\
\underline{21x-28}
\end{array}
$$

$\therefore$ H. C. F. $= 3x-4$.

$$
\begin{array}{r}
12x^2-31x+20 \\
\underline{2x+7} \\
24x^3-62x^2+40x \\
\underline{84x^2-217x+140}
\end{array}
$$

$\therefore$ L. C. M. $= 24x^3+22x^2-177x+140.$

4.

$$
\begin{array}{r}
12x^2-29x-8 \\
\underline{2} \\
8x^2+30x+7)\,24x^2-58x-16\,(3 \\
\underline{24x^2+90x+21} \\
-148x-37
\end{array}
$$

Dividing by -37,

$$
\begin{array}{r}
4x+1)\,8x^2+30x+7\,(2x+7 \\
\underline{8x^2+\ \ 2x} \\
28x+7 \\
\underline{28x+7}
\end{array}
$$

$\therefore$ H. C. F. $= 4x+1$.

$$
\begin{array}{r}
12x^2-29x-8 \\
\underline{2x+7} \\
24x^3-58x^2-16x \\
\underline{84x^2-203x-56}
\end{array}
$$

$\therefore$ L. C. M. $= 24x^3+26x^2-219x-56.$

5.

$$(6x^3 - 8x^2 - 30x) \div 2x = 3x^2 - 4x - 15.$$
$$(6ax^2 + 19ax + 15a) \div a = 6x^2 + 19x + 15.$$

$$\begin{array}{r} 3x^2 - 4x - 15)6x^2 + 19x + 15(2 \\ \underline{6x^2 - 8x - 30} \\ 27x + 45 \end{array}$$

Dividing by 9,

$$\begin{array}{r} 3x + 5)3x^2 - 4x - 15(x - 3 \\ \underline{3x^2 + 5x} \\ -9x - 15 \\ \underline{-9x - 15} \end{array}$$

$\therefore$ H. C. F. $= 3x + 5$.

$$\begin{array}{r} 6x^2 + 19x + 15 \\ \underline{x - 3} \\ 6x^3 + 19x^2 + 15x \\ \underline{-18x^2 - 57x - 45} \\ 6x^3 + x^2 - 42x - 45 \end{array}$$

$\therefore$ L. C. M. $= 2ax(6x^3 + x^2 - 42x - 45)$.

6.

$$\begin{array}{r} a^2 - 8ab + 7b^2)a^3 - 9a^2b + 23ab^2 - 15b^3(a - b \\ \underline{a^3 - 8a^2b + 7ab^2} \\ -a^2b + 16ab^2 - 15b^3 \\ \underline{-a^2b + 8ab^2 - 7b^3} \\ 8ab^2 - 8b^3 \end{array}$$

Dividing by $8b^2$,

$$\begin{array}{r} a - b)a^2 - 8ab + 7b^2(a - 7b \\ \underline{a^2 - ab} \\ -7ab + 7b^2 \\ \underline{-7ab + 7b^2} \end{array}$$

$\therefore$ H. C. F. $= a - b$.

$$\begin{array}{r} a^3 - 9a^2b + 23ab^2 - 15b^3 \\ \underline{a - 7b} \\ a^4 - 9a^3b + 23a^2b^2 - 15ab^3 \\ \underline{-7a^3b + 63a^2b^2 - 161ab^3 + 105b^4} \end{array}$$

$\therefore$ L. C. M. $= a^4 - 16a^3b + 86a^2b^2 - 176ab^3 + 105b^4$.

7.

$(2m^2n - 3mn - 2n) \div n = 2m^2 - 3m - 2.$

$$
\begin{array}{r}
2m^2 - 3m - 2)\,2m^4 - 6m^3 + 6m^2 - 8m + 8\,(m^2 \\
2m^4 - 3m^3 - 2m^2 \qquad\qquad\quad \\
\hline
-3m^3 + 8m^2 - 8m + 8 \\
2 \\
\hline
-6m^3 + 16m^2 - 16m + 16\,(-3m \\
-6m^3 + 9m^2 + 6m \qquad\quad \\
\hline
7m^2 - 22m + 16 \\
2 \\
\hline
14m^2 - 44m + 32\,(7 \\
14m^2 - 21m - 14 \\
\hline
-23m + 46
\end{array}
$$

Dividing by -23,

$$
\begin{array}{r}
m - 2)\,2m^2 - 3m - 2\,(2m + 1 \\
2m^2 - 4m \qquad \\
\hline
m - 2 \\
m - 2 \\
\hline
\end{array}
$$

$\therefore$ H. C. F. $= m - 2.$

$$
\begin{array}{r}
2m^4 - 6m^3 + 6m^2 - 8m + 8 \qquad \\
2m + 1 \qquad\qquad\qquad\qquad\qquad \\
\hline
4m^5 - 12m^4 + 12m^3 - 16m^2 + 16m \qquad \\
2m^4 - 6m^3 + 6m^2 - 8m + 8 \\
\hline
4m^5 - 10m^4 + 6m^3 - 10m^2 + 8m + 8
\end{array}
$$

$\therefore$ L. C. M. $= n(4m^5 - 10m^4 + 6m^3 - 10m^2 + 8m + 8)$
$= 2n(2m^5 - 5m^4 + 3m^3 - 5m^2 + 4m + 4).$

8.

$(6ax^2 - a^2x - 12a^3) \div a = 6x^2 - ax - 12a^2.$
$(10ax^2 - 17a^2x + 3a^3) \div a = 10x^2 - 17ax + 3a^2.$

$$
\begin{array}{r}
10x^2 - 17ax + 3a^2 \\
3 \\
\hline
6x^2 - ax - 12a^2)\,30x^2 - 51ax + 9a^2\,(5 \\
30x^2 - 5ax - 60a^2 \\
\hline
-46ax + 69a^2
\end{array}
$$

Dividing by $-23a$,

$$
\begin{array}{r}
2x - 3a)\,6x^2 - ax - 12a^2\,(3x + 4a \\
6x^2 - 9ax \qquad\quad \\
\hline
8ax - 12a^2 \\
8ax - 12a^2 \\
\hline
\end{array}
$$

$\therefore$ H. C. F. $= 2x - 3a.$

$$
\begin{array}{r}
10x^2 - 17ax + 3a^2 \qquad\qquad \\
3x + 4a \qquad\qquad\qquad\qquad \\
\hline
30x^3 - 51ax^2 + 9a^2x \qquad\qquad \\
40ax^2 - 68a^2x + 12a^3 \\
\hline
30x^3 - 11ax^2 - 59a^2x + 12a^3
\end{array}
$$

$\therefore$ L. C. M. $= a(30x^3 - 11ax^2 - 59a^2x + 12a^3).$

9.

$$a^3+a^2-8a-6)2a^3-5a^2-2a+2(2$$
$$\underline{2a^3+2a^2-16a-12}$$
$$-7a^2+14a+14$$

Dividing by -7, $$a^2-2a-2)a^3+a^2-8a-6(a+3$$
$$\underline{a^3-2a^2-2a}$$
$$3a^2-6a-6$$
$$\underline{3a^2-6a-6}$$

$\therefore$ H. C. F. $= a^2-2a-2$.

$$2a^3-5a^2-2a+2$$
$$\underline{a+3}$$
$$2a^4-5a^3-2a^2+2a$$
$$\underline{6a^3-15a^2-6a+6}$$

$\therefore$ L. C. M. $= 2a^4+a^3-17a^2-4a+6$.

10.

$$2x^3+x^2-x+3)2x^3+5x^2-x-6(1$$
$$\underline{2x^3+x^2-x+3}$$
$$4x^2-9$$

$$2x^3+x^2-x+3$$
$$\underline{2}$$
$$4x^2-9)4x^3+2x^2-2x+6(x$$
$$\underline{4x^3-9x}$$
$$2x^2+7x+6$$
$$\underline{2}$$
$$4x^2+14x+12(1$$
$$\underline{4x^2-9}$$
$$14x+21$$

Dividing by 7, $$2x+3)4x^2-9(2x-3$$
$$\underline{4x^2+6x}$$
$$-6x-9$$
$$\underline{-6x-9}$$

$\therefore$ H. C. F. $= 2x+3$.

$$2x+3)2x^3+x^2-x+3(x^2-x+1$$
$$\underline{2x^3+3x^2}$$
$$-2x^2-x+3$$
$$\underline{-2x^2-3x}$$
$$2x+3$$
$$\underline{2x+3}$$

$$2x^3+5x^2-x-6$$
$$\underline{x^2-x+1}$$
$$2x^5+5x^4-x^3-6x^2$$
$$-2x^4-5x^3+x^2+6x$$
$$\underline{2x^3+5x^2-x-6}$$

$\therefore$ L. C. M. $= 2x^5+3x^4-4x^3+5x-6$.

11.

$$
\begin{array}{l}
a^3 - 2a^2b + 2ab^2 - b^3)\,a^3 + a^2b - ab^2 - b^3\,(1 \\
\qquad\qquad\qquad\qquad\quad \underline{a^3 - 2a^2b + 2ab^2 - b^3} \\
\qquad\qquad\qquad\qquad\qquad\quad 3a^2b - 3ab^2
\end{array}
$$

Dividing by $3ab$,

$$
\begin{array}{l}
a - b)\,a^3 - 2a^2b + 2ab^2 - b^3\,(a^2 - ab + b^2 \\
\qquad\; \underline{a^3 - a^2b} \\
\qquad\; -a^2b + 2ab^2 - b^3 \\
\qquad\; \underline{-a^2b + ab^2} \\
\qquad\qquad\qquad\; ab^2 - b^3 \\
\qquad\qquad\qquad\; \underline{ab^2 - b^3}
\end{array}
$$

$\therefore$ H. C. F. $= a - b$.

$$
\begin{array}{l}
a^3 + a^2b - ab^2 - b^3 \\
\underline{a^2 - ab + b^2} \\
a^5 + a^4b - a^3b^2 - a^2b^3 \\
\quad - a^4b - a^3b^2 + a^2b^3 + ab^4 \\
\qquad\qquad\;\; \underline{a^3b^2 + a^2b^3 - ab^4 - b^5}
\end{array}
$$

$\therefore$ L. C. M. $= a^5 - a^3b^2 + a^2b^3 - b^5$.

12.

$(x^4 + 2x^3 + 2x^2 + x) \div x = x^3 + 2x^2 + 2x + 1.$

$(ax^3 - 2ax - a) \div a = x^3 - 2x - 1.$

$$
\begin{array}{l}
x^3 - 2x - 1)\,x^3 + 2x^2 + 2x + 1\,(1 \\
\qquad\qquad\quad \underline{x^3 \qquad\; - 2x - 1} \\
\qquad\qquad\qquad 2x^2 + 4x + 2
\end{array}
$$

Dividing by 2,

$$
\begin{array}{l}
x^2 + 2x + 1)\,x^3 \qquad - 2x - 1\,(x - 2 \\
\qquad\qquad\quad \underline{x^3 + 2x^2 + x} \\
\qquad\qquad\quad -2x^2 - 3x - 1 \\
\qquad\qquad\quad \underline{-2x^2 - 4x - 2} \\
\qquad\qquad\qquad\qquad\quad x + 1
\end{array}
$$

$$
\begin{array}{l}
x + 1)\,x^2 + 2x + 1\,(x + 1 \\
\qquad\; \underline{x^2 + x} \\
\qquad\qquad\;\; x + 1 \\
\qquad\qquad\;\; \underline{x + 1}
\end{array}
$$

$\therefore$ H. C. F. $= x + 1$.

$$
\begin{array}{l}
x + 1)\,x^3 \qquad - 2x - 1\,(x^2 - x - 1 \\
\qquad\; \underline{x^3 + x^2} \\
\qquad\; - x^2 - 2x - 1 \\
\qquad\; \underline{- x^2 - x} \\
\qquad\qquad\quad - x - 1 \\
\qquad\qquad\quad \underline{- x - 1}
\end{array}
$$

$$
\begin{array}{l}
x^3 + 2x^2 + 2x + 1 \\
\underline{x^2 - x - 1} \\
x^5 + 2x^4 + 2x^3 + x^2 \\
\quad - x^4 - 2x^3 - 2x^2 - x \\
\qquad\quad \underline{- x^3 - 2x^2 - 2x - 1} \\
x^5 + x^4 - x^3 - 3x^2 - 3x - 1
\end{array}
$$

$\therefore$ L. C. M. $= ax\,(x^5 + x^4 - x^3 - 3x^2 - 3x - 1)$.

13.

$$(2x^4-11x^3+3x^2+10x)\div x=2x^3-11x^2+3x+10.$$
$$(3x^4-14x^3-6x^2+\ 5x)\div x=3x^3-14x^2-6x+\ 5.$$

$$\begin{array}{r}
3x^3-14x^2-\ 6x+\ 5\\
2\\
\hline
2x^3-11x^2+3x+10)\,6x^3-28x^2-12x+10\,(3\\
6x^3-33x^2+\ 9x+30\\
\hline
5x^2-21x-20
\end{array}$$

$$\begin{array}{r}
2x^3-11x^2+\ \ 3x+\ 10\\
5\\
\hline
5x^2-21x-20)\,10x^3-55x^2+\ 15x+\ 50\,(2x\\
10x^3-42x^2-\ 40x\qquad\\
\hline
-13x^2+\ 55x+\ 50\\
5\\
\hline
-65x^2+275x+250\,(-13\\
-65x^2+273x+260\\
\hline
2x-\ 10
\end{array}$$

Dividing by 2,

$$\begin{array}{r}
x-5)\,5x^2-21x-20\,(5x+4\\
5x^2-25x\qquad\\
\hline
4x-20\\
4x-20\\
\hline
\end{array}$$

$\therefore$ H. C. F. $=x-5.$

$$\begin{array}{r}
x-5)\,2x^3-11x^2+3x+10\,(2x^2-x-2\\
2x^3-10x^2\qquad\qquad\\
\hline
-\quad x^2+3x+10\\
-\quad x^2+5x\qquad\\
\hline
-2x+10\\
-2x+10\\
\hline
\end{array}$$

$$\begin{array}{l}
3x^3-14x^2-\ 6x\ +\ 5\\
2x^2-\quad x\ -\ 2\\
\hline
6x^5-28x^4-12x^3+10x^2\\
\qquad-\ 3x^4+14x^3+\ 6x^2-\ 5x\\
\qquad\qquad\quad-\ 6x^3+28x^2+12x-10\\
\hline
6x^5-31x^4-\ 4x^3+44x^2+\ 7x-10
\end{array}$$

$\therefore$ L. C. M. $=x(6x^5-31x^4-4x^3+44x^2+7x-10).$

14.

$$\begin{array}{r}
x^4 - x^3 - 8x + 8)\,x^4 \qquad - 8x^2 + 9x - 2\,(1 \\
\underline{x^4 - x^3 \qquad - 8x + 8} \\
x^3 - 8x^2 + 17x - 10
\end{array}$$

$$\begin{array}{r}
x^3 - 8x^2 + 17x - 10)\,x^4 - x^3 \qquad - 8x + 8\,(x + 7 \\
\underline{x^4 - 8x^3 + 17x^2 - 10x \phantom{{}+8}} \\
7x^3 - 17x^2 + 2x + 8 \\
\underline{7x^3 - 56x^2 + 119x - 70} \\
39x^2 - 117x + 78
\end{array}$$

Dividing by 39,
$$\begin{array}{r}
x^2 - 3x + 2)\,x^3 - 8x^2 + 17x - 10\,(x - 5 \\
\underline{x^3 - 3x^2 + 2x \phantom{{}-10}} \\
-5x^2 + 15x - 10 \\
\underline{-5x^2 + 15x - 10}
\end{array}$$

$\therefore$ H. C. F. $= x^2 - 3x + 2.$

$$\begin{array}{r}
x^2 - 3x + 2)\,x^4 - x^3 \qquad - 8x + 8\,(x^2 + 2x + 4 \\
\underline{x^4 - 3x^3 + 2x^2 \phantom{{}-8x+8}} \\
2x^3 - 2x^2 - 8x + 8 \\
\underline{2x^3 - 6x^2 + 4x \phantom{{}+8}} \\
4x^2 - 12x + 8 \\
\underline{4x^2 - 12x + 8}
\end{array}$$

$$\begin{array}{lllllll}
x^4 - 8x^2 + 9x - 2 \\
\underline{x^2 + 2x + 4} \\
x^6 & & -8x^4 & +9x^3 & -2x^2 & & \\
 & 2x^5 & & -16x^3 & +18x^2 & -4x & \\
 & & 4x^4 & & -32x^2 & +36x & -8 \\
\hline
\end{array}$$

$\therefore$ L. C. M. $= x^6 + 2x^5 - 4x^4 - 7x^3 - 16x^2 + 32x - 8.$

Art. 140.—Page 79.

1.

$$x^2-1)2x^2-9x+7(2$$
$$\begin{array}{r} 2x^2 \qquad -2 \\ \hline -9x+9 \end{array}$$

Dividing by -9,

$$x-1)x^2 \qquad -1(x+1$$
$$\begin{array}{r} x^2-x \qquad \\ \hline x-1 \\ x-1 \\ \hline \end{array}$$

$$\begin{array}{r} 2x^2-9x+7 \qquad \\ x+1 \qquad \\ \hline 2x^3-9x^2+7x \qquad \\ 2x^2-9x+7 \\ \hline 2x^3-7x^2-2x+7 \end{array}$$

$$2x^2+3x-5)2x^3-7x^2-2x+7(x-5$$
$$\begin{array}{r} 2x^3+3x^2-5x \qquad \\ \hline -10x^2+3x+7 \\ -10x^2-15x+25 \\ \hline 18x-18 \end{array}$$

Dividing by 18,

$$x-1)2x^2+3x-5(2x+5$$
$$\begin{array}{r} 2x^2-2x \qquad \\ \hline 5x-5 \\ 5x-5 \\ \hline \end{array}$$

$$\begin{array}{r} 2x^3-7x^2-2x+7 \qquad \\ 2x+5 \qquad \qquad \qquad \\ \hline 4x^4-14x^3-4x^2+14x \qquad \\ 10x^3-35x^2-10x+35 \\ \hline \end{array}$$

$$\therefore \text{L. C. M.} = 4x^4-4x^3-39x^2+4x+35.$$

2.

$$
\begin{array}{r}
3a^2-2a-1)\,6a^2-a-1\,(2 \\
\underline{6a^2-4a-2} \\
3a+1
\end{array}
$$

$$
\begin{array}{r}
3a+1)\,3a^2-2a-1\,(a-1 \\
\underline{3a^2+a} \\
-3a-1 \\
\underline{-3a-1}
\end{array}
$$

$$
\begin{array}{r}
6a^2-a-1 \\
\underline{a-1} \\
6a^3-a^2-a \\
\underline{-6a^2+a+1} \\
6a^3-7a^2+1
\end{array}
$$

$$
\begin{array}{r}
6a^3-7a^2+1 \\
\underline{3} \\
9a^2-3a-2)\,18a^3-21a^2+3\,(2a \\
\underline{18a^3-6a^2-4a} \\
-15a^2+4a+3 \\
\underline{3} \\
-45a^2+12a+9\,(-5 \\
\underline{-45a^2+15a+10} \\
-3a-1
\end{array}
$$

Changing signs,

$$
\begin{array}{r}
3a+1)\,9a^2-3a-2\,(3a-2 \\
\underline{9a^2+3a} \\
-6a-2 \\
\underline{-6a-2}
\end{array}
$$

$$
\begin{array}{r}
6a^3-7a^2+1 \\
\underline{3a-2} \\
18a^4-21a^3+3a \\
\underline{-12a^3+14a^2-2}
\end{array}
$$

$$\therefore \text{L.C.M.} = 18a^4-33a^3+14a^2+3a-2.$$

3.

$$\begin{array}{r}
2x^2-5x+2)4x^2+\ \ 4x-3(2 \\
4x^2-10x+4 \\ \hline
14x-7
\end{array}$$

Dividing by 7,

$$\begin{array}{r}
2x-1)2x^2-5x+2(x-2 \\
2x^2-\ \ x \\ \hline
-4x+2 \\
-4x+2 \\ \hline
\end{array}$$

$$\begin{array}{r}
4x^2+4x-\ \ 3 \\
x\ -2 \\ \hline
4x^3+4x^2-\ \ 3x \\
-8x^2-\ \ 8x+6 \\ \hline
4x^3-4x^2-11x+6
\end{array}$$

$$\begin{array}{r}
4x^3-\ \ 4x^2-\ \ 11x+\ \ 6 \\
5 \\ \hline
10x^2-7x+1)20x^3-20x^2-\ \ 55x+\ \ 30(2x \\
20x^3-14x^2+\ \ \ 2x \\ \hline
-\ \ 6x^2-\ \ 57x+\ \ 30 \\
5 \\ \hline
-30x^2-285x+150(-3 \\
-30x^2+\ \ 21x-\ \ \ 3 \\ \hline
-306x+153
\end{array}$$

Dividing by -153,

$$\begin{array}{r}
2x-1)10x^2-7x+1(5x-1 \\
10x^2-5x \\ \hline
-2x+1 \\
-2x+1 \\ \hline
\end{array}$$

$$\begin{array}{r}
4x^3-\ \ 4x^2-11x+\ 6 \\
5x\ -\ \ 1 \\ \hline
20x^4-20x^3-55x^2+30x \\
-\ \ 4x^3+\ \ 4x^2+11x-6 \\ \hline
\end{array}$$

$\therefore$ L. C. M. $= 20x^4-24x^3-51x^2+41x-6.$

4.

$$(4x^2-6x-18)\div 2=2x^2-3x-9.$$
$$(4x^3+4x^2-3x)\div x=4x^2+4x-3.$$
$$(6x^4+5x^3-6x^2)\div x^2=6x^2+5x-6.$$

$$\begin{array}{r}
2x^2-3x-9)\,4x^2+4x-3\,(2\\
\underline{4x^2-6x-18}\\
10x+15
\end{array}$$

Dividing by 5,
$$\begin{array}{r}
2x+3)\,2x^2-3x-9\,(x-3\\
\underline{2x^2+3x}\\
-6x-9\\
\underline{-6x-9}
\end{array}$$

$$\begin{array}{r}
4x^2+4x-3\\
\underline{x-3}\\
4x^3+4x^2-3x\\
\underline{-12x^2-12x+9}\\
4x^3-8x^2-15x+9
\end{array}$$

$$\begin{array}{r}
4x^3-8x^2-15x+9\\
\underline{3}\\
6x^2+5x-6)\,12x^3-24x^2-45x+27\,(2x\\
\underline{12x^3+10x^2-12x}\\
-34x^2-33x+27\\
\underline{3}\\
-102x^2-99x+81\,(-17\\
\underline{-102x^2-85x+102}\\
-14x-21
\end{array}$$

Dividing by -14,
$$\begin{array}{r}
2x+3)\,6x^2+5x-6\,(3x-2\\
\underline{6x^2+9x}\\
-4x-6\\
\underline{-4x-6}
\end{array}$$

$$\begin{array}{r}
4x^3-8x^2-15x+9\\
\underline{3x-2}\\
12x^4-24x^3-45x^2+27x\\
\underline{-8x^3+16x^2+30x-18}\\
12x^4-32x^3-29x^2+57x-18
\end{array}$$

$\therefore$ L.C.M. $=2x^2(12x^4-32x^3-29x^2+57x-18)$.

5.

$$
\begin{array}{r}
a^3-6a^2+11a-6)\,a^3-a^2-14a+24\,(1 \\
\underline{a^3-6a^2+11a-6} \\
5a^2-25a+30
\end{array}
$$

Dividing by 5,

$$
\begin{array}{r}
a^2-5a+6)\,a^3-6a^2+11a-6\,(a-1 \\
\underline{a^3-5a^2+6a} \\
-a^2+5a-6 \\
\underline{-a^2+5a-6}
\end{array}
$$

$$
\begin{array}{l}
a^3-a^2-14a+24 \\
\underline{a-1} \\
a^4-a^3-14a^2+24a \\
\underline{-a^3+a^2+14a-24} \\
a^4-2a^3-13a^2+38a-24
\end{array}
$$

$$
\begin{array}{r}
a^3+a^2-17a+15)\,a^4-2a^3-13a^2+38a-24\,(a-3 \\
\underline{a^4+a^3-17a^2+15a} \\
-3a^3+4a^2+23a-24 \\
\underline{-3a^3-3a^2+51a-45} \\
7a^2-28a+21
\end{array}
$$

Dividing by 7,

$$
\begin{array}{r}
a^2-4a+3)\,a^3+a^2-17a+15\,(a+5 \\
\underline{a^3-4a^2+3a} \\
5a^2-20a+15 \\
\underline{5a^2-20a+15}
\end{array}
$$

$$
\begin{array}{l}
a^4-2a^3-13a^2+38a-24 \\
\underline{a+5} \\
a^5-2a^4-13a^3+38a^2-24a \\
\underline{5a^4-10a^3-65a^2+190a-120}
\end{array}
$$

$\therefore$ L.C.M. $= a^5+3a^4-23a^3-27a^2+166a-120.$

CHAPTER XI.

Art. 149.—Pages 83, 84.

3. $\dfrac{x^4y^2z}{xy^2z^3}=\dfrac{x^3}{z^2}.$

4. $\dfrac{2a^2b^5c}{5a^3bc^3}=\dfrac{2b^4}{5ac^2}.$

5. $\dfrac{12xy^2}{32x^3}=\dfrac{2^2\cdot 3\cdot xy^2}{2^5\cdot x^3}=\dfrac{3\cdot y^2}{2^3\cdot x^2}=\dfrac{3y^2}{8x^2}.$

6. $\dfrac{32mn}{56m^4n^3}=\dfrac{2^5\cdot mn}{2^3\cdot 7\cdot m^4n^3}=\dfrac{2^2}{7m^3n^2}=\dfrac{4}{7m^3n^2}.$

7. $\dfrac{65x^2y^3z^4}{26x^4y^3z^2}=\dfrac{5\cdot 13\cdot x^2y^3z^4}{2\cdot 13\cdot x^4y^3z^2}=\dfrac{5z^2}{2x^2}.$

8. $\dfrac{54a^3c^2}{72a^2bc}=\dfrac{2\cdot 3^3\cdot a^3c^2}{2^3\cdot 3^2\cdot a^2bc}=\dfrac{3\cdot ac}{2^2\cdot b}=\dfrac{3ac}{4b}.$

9. $\dfrac{15mxy^2}{75mx^2y^3}=\dfrac{3\cdot 5\cdot mxy^2}{3\cdot 5^2\cdot mx^2y^3}=\dfrac{1}{5xy}.$

10. $\dfrac{115c^3x^2y}{23c^2x^2}=\dfrac{5\cdot 23\cdot c^3x^2y}{23\cdot c^2x^2}=5cy.$

11. $\dfrac{154m^2x^3}{88m^3xy^2}=\dfrac{2\cdot 7\cdot 11\cdot m^2x^3}{2^3\cdot 11\cdot m^3xy^2}=\dfrac{7\cdot x^2}{2^2\cdot my^2}=\dfrac{7x^2}{4my^2}.$

12. $\dfrac{2a^2cd+2abcd}{6a^2xy+6abxy}=\dfrac{2\cdot acd\,(a+b)}{2\cdot 3\cdot axy\,(a+b)}=\dfrac{cd}{3xy}.$

13. $\dfrac{3x^5-6x^4y}{6x^2y^2-12xy^3}=\dfrac{3\cdot x^4\,(x-2y)}{2\cdot 3\cdot xy^2\,(x-2y)}=\dfrac{x^3}{2y^2}.$

14. $\dfrac{2x^3y-6x^2y}{x^2-8x+15}=\dfrac{2x^2y\,(x-3)}{(x-5)(x-3)}=\dfrac{2x^2y}{x-5}.$

15. $\dfrac{a^2-2a-15}{a^2+10a+21}=\dfrac{(a+3)(a-5)}{(a+3)(a+7)}=\dfrac{a-5}{a+7}.$

16. $\dfrac{6a^3b+3a^2b^2}{3a^2b^2+6ab^3}=\dfrac{3a^2b\,(2a+b)}{3ab^2\,(a+2b)}=\dfrac{a\,(2a+b)}{b\,(a+2b)}=\dfrac{2a^2+ab}{ab+2b^2}.$

17. $\dfrac{4c^2-20c+25}{4c^3-25c}=\dfrac{(2c-5)^2}{c\,(2c+5)(2c-5)}=\dfrac{2c-5}{c\,(2c+5)}.$

18. $\dfrac{m^2-10m+16}{m^2+m-72}=\dfrac{(m-8)(m-2)}{(m-8)(m+9)}=\dfrac{m-2}{m+9}.$

19. $\dfrac{9an^2-4a}{9bn^2-12bn+4b}=\dfrac{a(3n+2)(3n-2)}{b(3n-2)^2}=\dfrac{a(3n+2)}{b(3n-2)}.$

20. $\dfrac{a^2-4b^2}{a^2+ab-6b^2}=\dfrac{(a+2b)(a-2b)}{(a+3b)(a-2b)}=\dfrac{a+2b}{a+3b}.$

21. $\dfrac{8x^3+y^3}{4x^3-2x^2y+xy^2}=\dfrac{(2x+y)(4x^2-2xy+y^2)}{x(4x^2-2xy+y^2)}=\dfrac{2x+y}{x}.$

22. $\dfrac{ac-ad-bc+bd}{a^3-b^3}=\dfrac{(a-b)(c-d)}{(a-b)(a^2+ab+b^2)}=\dfrac{c-d}{a^2+ab+b^2}.$

23. $\dfrac{ax^2-4a}{x^3-9x^2+14x}=\dfrac{a(x+2)(x-2)}{x(x-7)(x-2)}=\dfrac{a(x+2)}{x(x-7)}.$

24. $\dfrac{27y^3-125}{9y^2-30y+25}=\dfrac{(3y-5)(9y^2+15y+25)}{(3y-5)^2}=\dfrac{9y^2+15y+25}{3y-5}.$

25. $\dfrac{x^3-x^2+2x-2}{2x^3+x^2+4x+2}=\dfrac{(x^2+2)(x-1)}{(x^2+2)(2x+1)}=\dfrac{x-1}{2x+1}.$

26. $\dfrac{x^2-4x+16}{ax^4+64ax}=\dfrac{x^2-4x+16}{ax(x+4)(x^2-4x+16)}=\dfrac{1}{ax(x+4)}.$

27. $\dfrac{a^2-(b+c)^2}{(a-b)^2-c^2}=\dfrac{(a+b+c)(a-b-c)}{(a-b+c)(a-b-c)}=\dfrac{a+b+c}{a-b+c}.$

28. $\dfrac{(x^2-4)(x^2-3x+2)}{(x^2-4x+4)(x^2+x-2)}=\dfrac{(x+2)(x-2)(x-1)(x-2)}{(x-2)^2(x-1)(x+2)}=1.$

29. $\dfrac{(a-b)^2-(c-d)^2}{(a-c)^2-(b-d)^2}=\dfrac{(a-b+c-d)(a-b-c+d)}{(a+b-c-d)(a-b-c+d)}=\dfrac{a-b+c-d}{a+b-c-d}.$

Art. 150.—Page 85.

2.

$$
\begin{array}{r}
x^2-6x+5)\,3x^2+4x-7\,(3 \\
\underline{3x^2-18x+15} \\
22x-22
\end{array}
$$

Dividing by 22,

$$
\begin{array}{r}
x-1)\,x^2-6x+5\,(x-5 \\
\underline{x^2-x} \\
-5x+5 \\
\underline{-5x+5}
\end{array}
$$

$\therefore$ H.C.F. $=x-1$.

$$
\begin{array}{r}
x-1)\,3x^2+4x-7\,(3x+7 \\
\underline{3x^2-3x} \\
7x-7 \\
\underline{7x-7}
\end{array}
$$

$$\therefore \frac{x^2-6x+5}{3x^2+4x-7}=\frac{x-5}{3x+7}.$$

3.

$$2a^2-7a+6)10a^2-a-21(5$$
$$\underline{10a^2-35a+30}$$
$$34a-51$$

Dividing by 17, $\quad 2a-3)2a^2-7a+6(a-2$
$$\underline{2a^2-3a}$$
$$-4a+6$$
$$\underline{-4a+6}$$

$\therefore$ H.C.F. $=2a-3$.

$$2a-3)10a^2-a-21(5a+7$$
$$\underline{10a^2-15a}$$
$$14a-21$$
$$\underline{14a-21}$$

$$\therefore \frac{10a^2-a-21}{2a^2-7a+6}=\frac{5a+7}{a-2}.$$

4.

$$2m^2-5m+3)12m^2-28m+15(6$$
$$\underline{12m^2-30m+18}$$
$$2m-3$$

$$2m-3)2m^2-5m+3(m-1$$
$$\underline{2m^2-3m}$$
$$-2m+3$$
$$\underline{-2m+3}$$

$\therefore$ H.C.F. $=2m-3$.

$$2m-3)12m^2-28m+15(6m-5$$
$$\underline{12m^2-18m}$$
$$-10m+15$$
$$\underline{-10m+15}$$

$$\therefore \frac{2m^2-5m+3}{12m^2-28m+15}=\frac{m-1}{6m-5}.$$

5.

$$x^2-2x-3)x^3-2x^2-2x-3(x$$
$$\underline{x^3-2x^2-3x}$$
$$x-3$$

$$x-3)x^2-2x-3(x+1$$
$$\underline{x^2-3x}$$
$$x-3$$
$$\underline{x-3}$$

$\therefore$ H. C. F. $=x-3.$

$$x-3)x^3-2x^2-2x-3(x^2+x+1$$
$$\underline{x^3-3x^2}$$
$$x^2-2x-3$$
$$\underline{x^2-3x}$$
$$x-3$$
$$\underline{x-3}$$

$$\therefore \frac{x^2-2x-3}{x^3-2x^2-2x-3}=\frac{x+1}{x^2+x+1}.$$

6.

$$12m^2+16mn-3n^2$$
$$\underline{5}$$
$$10m^2+mn-21n^2)60m^2+80mn-15n^2(6$$
$$\underline{60m^2+6mn-126n^2}$$
$$74mn+111n^2$$

Dividing by $37n$, $\quad 2m+3n)10m^2+mn-21n^2(5m-7n$
$$\underline{10m^2+15mn}$$
$$-14mn-21n^2$$
$$\underline{-14mn-21n^2}$$

$\therefore$ H. C. F. $=2m+3n.$

$$2m+3n)12m^2+16mn-3n^2(6m-n$$
$$\underline{12m^2+18mn}$$
$$-2mn-3n^2$$
$$\underline{-2mn-3n^2}$$

$$\therefore \frac{12m^2+16mn-3n^2}{10m^2+mn-21n^2}=\frac{6m-n}{5m-7n}.$$

7.

$$\begin{array}{r}
x^3+x^2-3x-2)x^3-4x^2+2x+3(1 \\
\underline{x^3+x^2-3x-2} \\
-5x^2+5x+5
\end{array}$$

Dividing by -5,

$$\begin{array}{r}
x^2-x-1)x^3+x^2-3x-2(x+2 \\
\underline{x^3-x^2-x} \\
2x^2-2x-2 \\
\underline{2x^2-2x-2}
\end{array}$$

$\therefore$ H.C.F. $=x^2-x-1$.

$$\begin{array}{r}
x^2-x-1)x^3-4x^2+2x+3(x-3 \\
\underline{x^3-x^2-x} \\
-3x^2+3x+3 \\
\underline{-3x^2+3x+3}
\end{array}$$

$$\therefore \frac{x^3+x^2-3x-2}{x^3-4x^2+2x+3}=\frac{x+2}{x-3}.$$

8.

$$\begin{array}{r}
2x^3+5x^2-2x+3)6x^3-7x^2+5x-2(3 \\
\underline{6x^3+15x^2-6x+9} \\
-22x^2+11x-11
\end{array}$$

Dividing by -11,

$$\begin{array}{r}
2x^2-x+1)2x^3+5x^2-2x+3(x+3 \\
\underline{2x^3-x^2+x} \\
6x^2-3x+3 \\
\underline{6x^2-3x+3}
\end{array}$$

$\therefore$ H.C.F. $=2x^2-x+1$.

$$\begin{array}{r}
2x^2-x+1)6x^3-7x^2+5x-2(3x-2 \\
\underline{6x^3-3x^2+3x} \\
-4x^2+2x-2 \\
\underline{-4x^2+2x-2}
\end{array}$$

$$\therefore \frac{6x^3-7x^2+5x-2}{2x^3+5x^2-2x+3}=\frac{3x-2}{x+3}.$$

9.

$$\begin{array}{r}
6y^3-19y^2+7y+12)6y^3-25y^2+17y+20(1 \\
\underline{6y^3-19y^2+7y+12} \\
-6y^2+10y+8
\end{array}$$

Dividing by -2,

$$\begin{array}{l}
3y^2-5y-4)6y^3-19y^2+7y+12(2y-3 \\
\qquad\qquad\quad \underline{6y^3-10y^2-8y} \\
\qquad\qquad\qquad\quad -9y^2+15y+12 \\
\qquad\qquad\qquad\quad \underline{-9y^2+15y+12}
\end{array}$$

$\therefore$ H. C. F. $=3y^2-5y-4$.

$$\begin{array}{l}
3y^2-5y-4)6y^3-25y^2+17y+20(2y-5 \\
\qquad\qquad\quad \underline{6y^3-10y^2-8y} \\
\qquad\qquad\qquad\quad -15y^2+25y+20 \\
\qquad\qquad\qquad\quad \underline{-15y^2+25y+20}
\end{array}$$

$$\therefore \frac{6y^3-19y^2+7y+12}{6y^3-25y^2+17y+20}=\frac{2y-3}{2y-5}.$$

10.

$$\begin{array}{l}
a^3-3a^2+a+2)2a^3-3a^2-a-2(2 \\
\qquad\qquad\quad \underline{2a^3-6a^2+2a+4} \\
\qquad\qquad\qquad\quad 3a^2-3a-6
\end{array}$$

Dividing by 3,

$$\begin{array}{l}
a^2-a-2)a^3-3a^2+a+2(a-2 \\
\qquad\qquad \underline{a^3-a^2-2a} \\
\qquad\qquad\quad -2a^2+3a+2 \\
\qquad\qquad\quad \underline{-2a^2+2a+4} \\
\qquad\qquad\qquad\qquad a-2
\end{array}$$

$$\begin{array}{l}
a-2)a^2-a-2(a+1 \\
\qquad \underline{a^2-2a} \\
\qquad\qquad a-2 \\
\qquad\qquad \underline{a-2}
\end{array}$$

$\therefore$ H. C. F. $=a-2$.

$$\begin{array}{l}
a-2)a^3-3a^2+a+2(a^2-a-1 \\
\qquad \underline{a^3-2a^2} \\
\qquad\quad -a^2+a+2 \\
\qquad\quad \underline{-a^2+2a} \\
\qquad\qquad\quad -a+2 \\
\qquad\qquad\quad \underline{-a+2}
\end{array}$$

$$\begin{array}{l}
a-2)2a^3-3a^2-a-2(2a^2+a+1 \\
\qquad \underline{2a^3-4a^2} \\
\qquad\qquad a^2-a-2 \\
\qquad\qquad \underline{a^2-2a} \\
\qquad\qquad\quad a-2 \\
\qquad\qquad\quad \underline{a-2}
\end{array}$$

$$\therefore \frac{a^3-3a^2+a+2}{2a^3-3a^2-a-2}=\frac{a^2-a-1}{2a^2+a+1}.$$

11.

$$
\begin{array}{l}
x^3-2x^2y+4xy^2-3y^3)\,x^3-4x^2y+4xy^2-y^3\,(1 \\
\qquad x^3-2x^2y+4xy^2-3y^3 \\
\hline
\qquad -2x^2y \qquad -2y^3
\end{array}
$$

Dividing by $-2y$,

$$
\begin{array}{l}
x^2-y^2)\,x^3-2x^2y+4xy^2-3y^3\,(x-2y \\
\qquad x^3 \qquad -xy^2 \\
\hline
\qquad -2x^2y+5xy^2-3y^3 \\
\qquad -2x^2y \qquad +2y^3 \\
\hline
\qquad 5xy^2-5y^3
\end{array}
$$

Dividing by $5y^2$,

$$
\begin{array}{l}
x-y)\,x^2-y^2\,(x+y \\
\qquad x^2-xy \\
\hline
\qquad xy-y^2 \\
\qquad xy-y^2 \\
\hline
\end{array}
$$

$\therefore$ H.C.F. $= x-y$.

$$
\begin{array}{l}
x-y)\,x^3-4x^2y+4xy^2-y^3\,(x^2-3xy+y^2 \\
\qquad x^3-x^2y \\
\hline
\qquad -3x^2y+4xy^2-y^3 \\
\qquad -3x^2y+3xy^2 \\
\hline
\qquad xy^2-y^3 \\
\qquad xy^2-y^3 \\
\hline
\end{array}
$$

$$
\begin{array}{l}
x-y)\,x^3-2x^2y+4xy^2-3y^3\,(x^2-xy+3y^2 \\
\qquad x^3-x^2y \\
\hline
\qquad -x^2y+4xy^2-3y^3 \\
\qquad -x^2y+xy^2 \\
\hline
\qquad 3xy^2-3y^3 \\
\qquad 3xy^2-3y^3 \\
\hline
\end{array}
$$

$$\therefore \frac{x^3-4x^2y+4xy^2-y^3}{x^3-2x^2y+4xy^2-3y^3}=\frac{x^2-3xy+y^2}{x^2-xy+3y^2}.$$

Art. 155.—Pages 87, 88.

3. $\dfrac{5x^2-10x+4}{5x}=x-2+\dfrac{4}{5x}.$ 4. $\dfrac{6x^3-3x^2+9x-7}{3x}=2x^2-x+3-\dfrac{7}{3x}.$

5.

$$
\begin{array}{l}
x+y)\,x^3+2y^3\,(x^2-xy+y^2+\dfrac{y^3}{x+y} \\
\quad\;\; \underline{x^3+x^2y} \\
\qquad\; -x^2y+2y^3 \\
\qquad\; \underline{-x^2y-xy^2} \\
\qquad\qquad\quad xy^2+2y^3 \\
\qquad\qquad\quad \underline{xy^2+\;\;y^3} \\
\qquad\qquad\qquad\quad y^3
\end{array}
$$

6.

$$
\begin{array}{l}
x-3)\,2x^2\qquad -41\,(2x+6-\dfrac{23}{x-3} \\
\qquad\; \underline{2x^2-6x} \\
\qquad\qquad\;\; 6x-41 \\
\qquad\qquad\;\; \underline{6x-18} \\
\qquad\qquad\qquad -23
\end{array}
$$

7.

$$
\begin{array}{l}
a^2+a-1)\,a^3-\;a^2-\;a-2\,(a-2+\dfrac{2a-4}{a^2+a-1} \\
\qquad\qquad \underline{a^3+\;a^2-\;a} \\
\qquad\qquad\; -2a^2\qquad -2 \\
\qquad\qquad\; \underline{-2a^2-2a+2} \\
\qquad\qquad\qquad\quad 2a-4
\end{array}
$$

8.

$$
\begin{array}{l}
4x-1)\,12x^2-8x+7\,(3x+\dfrac{-5x+7}{4x-1}=3x-\dfrac{5x-7}{4x-1} \\
\qquad\quad \underline{12x^2-3x} \\
\qquad\qquad\quad -5x+7
\end{array}
$$

9.

$$
\begin{array}{l}
a+b)\,a^4+\;\;b^4\,(a^3-a^2b+ab^2-b^3+\dfrac{2b^4}{a+b} \\
\quad\;\; \underline{a^4+a^3b} \\
\qquad\; -a^3b+\;\;b^4 \\
\qquad\; \underline{-a^3b-a^2b^2} \\
\qquad\qquad\quad a^2b^2+\;\;b^4 \\
\qquad\qquad\quad \underline{a^2b^2+ab^3} \\
\qquad\qquad\qquad\; -ab^3+\;\;b^4 \\
\qquad\qquad\qquad\; \underline{-ab^3-\;\;b^4} \\
\qquad\qquad\qquad\qquad\quad 2b^4
\end{array}
$$

10.

$$
\begin{array}{l}
2m-3n)4m^3-16m^2n+29mn^2-22n^3(2m^2-5mn+7n^2-\dfrac{n^3}{2m-3n} \\
\qquad\quad \underline{4m^3-6m^2n} \\
\qquad\qquad -10m^2n+29mn^2-22n^3 \\
\qquad\qquad \underline{-10m^2n+15mn^2} \\
\qquad\qquad\qquad\qquad 14mn^2-22n^3 \\
\qquad\qquad\qquad\qquad \underline{14mn^2-21n^3} \\
\qquad\qquad\qquad\qquad\qquad -\ n^3
\end{array}
$$

11.

$$
\begin{array}{l}
2a^2-a-3)2a^4-a^3-9a^2+14(a^2-3+\dfrac{-3a+5}{2a^2-a-3}=a^2-3-\dfrac{3a-5}{2a^2-a-3} \\
\qquad\quad \underline{2a^4-a^3-3a^2} \\
\qquad\qquad\quad -6a^2\qquad+14 \\
\qquad\qquad\quad \underline{-6a^2+3a+9} \\
\qquad\qquad\qquad\quad -3a+5
\end{array}
$$

12.

$$
\begin{array}{l}
x^2+x+1)x^3+2x^2+3x+4(x+1+\dfrac{x+3}{x^2+x+1} \\
\qquad\quad \underline{x^3+x^2+x} \\
\qquad\qquad\quad x^2+2x+4 \\
\qquad\qquad\quad \underline{x^2+x+1} \\
\qquad\qquad\qquad\quad x+3
\end{array}
$$

13.

$$
\begin{array}{l}
x+y)x^5-y^5(x^4-x^3y+x^2y^2-xy^3+y^4-\dfrac{2y^5}{x+y} \\
\qquad \underline{x^5+x^4y} \\
\qquad -x^4y-y^5 \\
\qquad \underline{-x^4y-x^3y^2} \\
\qquad\qquad\quad x^3y^2-y^5 \\
\qquad\qquad\quad \underline{x^3y^2+x^2y^3} \\
\qquad\qquad\qquad -x^2y^3-y^5 \\
\qquad\qquad\qquad \underline{-x^2y^3-xy^4} \\
\qquad\qquad\qquad\qquad\quad xy^4-y^5 \\
\qquad\qquad\qquad\qquad\quad \underline{xy^4+y^5} \\
\qquad\qquad\qquad\qquad\qquad -2y^5
\end{array}
$$

14.

$$
\begin{array}{l}
3x^2-2x+1)6x^3-13x^2+6x-6(2x-3+\dfrac{-2x-3}{3x^2-2x+1}=2x-3-\dfrac{2x+3}{3x^2-2x+1} \\
\qquad\qquad \underline{6x^3-4x^2+2x} \\
\qquad\qquad\quad -\ 9x^2+4x-6 \\
\qquad\qquad\quad \underline{-\ 9x^2+6x-3} \\
\qquad\qquad\qquad\qquad -2x-3
\end{array}
$$

Art. 156.—Page 89.

3. $x+1+\dfrac{x+1}{x}=\dfrac{x(x+1)+x+1}{x}=\dfrac{x^2+x+x+1}{x}=\dfrac{x^2+2x+1}{x}=\dfrac{(x+1)^2}{x}.$

4. $x+1-\dfrac{4}{x+3}=\dfrac{(x+1)(x+3)-4}{x+3}=\dfrac{x^2+4x+3-4}{x+3}=\dfrac{x^2+4x-1}{x+3}.$

5. $\dfrac{2m^2-3n^2}{3m+n}+m-n=\dfrac{2m^2-3n^2+(m-n)(3m+n)}{3m+n}$

$=\dfrac{2m^2-3n^2+3m^2-2mn-n^2}{3m+n}=\dfrac{5m^2-2mn-4n^2}{3m+n}.$

6. $7x-3-\dfrac{53x-20}{8}=\dfrac{56x-24-(53x-20)}{8}=\dfrac{56x-24-53x+20}{8}$

$=\dfrac{3x-4}{8}.$

7. $1-\dfrac{m-n}{m+n}=\dfrac{m+n-(m-n)}{m+n}=\dfrac{m+n-m+n}{m+n}=\dfrac{2n}{m+n}.$

8. $a+b-\dfrac{a^2+b^2}{a+b}=\dfrac{(a+b)^2-(a^2+b^2)}{a+b}=\dfrac{a^2+2ab+b^2-a^2-b^2}{a+b}=\dfrac{2ab}{a+b}.$

9. $\dfrac{2}{2x+1}+3x-2=\dfrac{2+(2x+1)(3x-2)}{2x+1}=\dfrac{2+6x^2-x-2}{2x+1}=\dfrac{6x^2-x}{2x+1}.$

10. $a^2-b^2+\dfrac{ab(a+b)}{a-b}=\dfrac{(a-b)(a^2-b^2)+ab(a+b)}{a-b}$

$=\dfrac{a^3-a^2b-ab^2+b^3+a^2b+ab^2}{a-b}=\dfrac{a^3+b^3}{a-b}.$

11. $\dfrac{x+y}{x-y}-1=\dfrac{x+y-(x-y)}{x-y}=\dfrac{x+y-x+y}{x-y}=\dfrac{2y}{x-y}.$

12. $m-n+\dfrac{m^2+n^2}{m+n}=\dfrac{(m+n)(m-n)+m^2+n^2}{m+n}=\dfrac{m^2-n^2+m^2+n^2}{m+n}=\dfrac{2m^2}{m+n}.$

13. $a^2-ab+b^2-\dfrac{2b^3}{a+b}=\dfrac{(a+b)(a^2-ab+b^2)-2b^3}{a+b}$

$=\dfrac{a^3+b^3-2b^3}{a+b}=\dfrac{a^3-b^3}{a+b}.$

14. $x^2-3x-\dfrac{2x(3-x)}{x-2}=\dfrac{(x-2)(x^2-3x)-2x(3-x)}{x-2}$

$=\dfrac{x^3-5x^2+6x-6x+2x^2}{x-2}=\dfrac{x^3-3x^2}{x-2}.$

15. $\dfrac{m^3+n^3}{m^2+mn+n^2}-(m-n)=\dfrac{m^3+n^3-(m-n)(m^2+mn+n^2)}{m^2+mn+n^2}$

$$=\frac{m^3+n^3-(m^3-n^3)}{m^2+mn+n^2}=\frac{m^3+n^3-m^3+n^3}{m^2+mn+n^2}=\frac{2n^3}{m^2+mn+n^2}.$$

16. $1+2x+4x^2+\dfrac{x^3+1}{2x-1}=\dfrac{(2x-1)(4x^2+2x+1)+x^3+1}{2x-1}$

$$=\frac{8x^3-1+x^3+1}{2x-1}=\frac{9x^3}{2x-1}.$$

17. $x-2y-\dfrac{x^3-8y^3}{x^2-4xy+4y^2}=x-2y-\dfrac{(x-2y)(x^2+2xy+4y^2)}{(x-2y)^2}$

$$=x-2y-\frac{x^2+2xy+4y^2}{x-2y}=\frac{(x-2y)^2-(x^2+2xy+4y^2)}{x-2y}$$

$$=\frac{x^2-4xy+4y^2-x^2-2xy-4y^2}{x-2y}=\frac{-6xy}{x-2y}=\frac{6xy}{2y-x}.$$

18. $x^2-2x+3-\dfrac{x^3+13x-5}{x^2+3x-2}=\dfrac{(x^2-2x+3)(x^2+3x-2)-(x^3+13x-5)}{x^2+3x-2}$

$$=\frac{x^4+x^3-5x^2+13x-6-x^3-13x+5}{x^2+3x-2}=\frac{x^4-5x^2-1}{x^2+3x-2}.$$

Art. 157.—Pages 90, 91.

2. The L. C. M. of 14, 21, and 6 is 42. Multiplying the terms of the first fraction by 3, of the second by 2, and of the third by 7, we have:

$$\frac{9ab}{42},\quad \frac{4ac}{42},\quad \frac{35bc}{42}.$$

3. The L. C. M. of a^3x^2, ax^3, and a^2x is a^3x^3. Multiplying the terms of the first fraction by x, of the second by a^2, and of the third by ax^2, we have:

$$\frac{2x}{a^3x^3},\quad \frac{3a^2}{a^3x^3},\quad \frac{4ax^2}{a^3x^3}.$$

4. The L. C. M. of $8ab^2$ and $12a^2c$ is $24a^2b^2c$. Multiplying the terms of the first fraction by $3ac$, and of the second by $2b^2$, we have:

$$\frac{12ac^2-3ac}{24a^2b^2c},\quad \frac{6b^3-4b^2}{24a^2b^2c}.$$

5. The L. C. M. of $6x^2y$, $8y^2z$, and $10xz^2$ is $120x^2y^2z^2$. Multiplying the terms of the first fraction by $20yz^2$, of the second by $15x^2z$, and of the third by $12xy^2$, we have:

$$\frac{100ayz^3}{120x^2y^2z^2}, \quad \frac{45bx^3z}{120x^2y^2z^2}, \quad \frac{84cxy^3-12mxy^2}{120x^2y^2z^2}.$$

6. $$\begin{aligned} a^2+a-6 &= (a+3)\qquad\quad(a-2)\\ a^2-4 &= \underline{\qquad\quad(a+2)(a-2)} \end{aligned}$$

$\therefore$ L. C. D. $=(a+3)(a+2)(a-2)=(a+3)(a^2-4)$.

Multiplying the terms of the first fraction by $a+2$, and of the second by $a+3$, we have:

$$\frac{2a^2+4a}{(a+3)(a^2-4)}, \quad \frac{4a^2+12a}{(a+3)(a^2-4)}.$$

7. $$\begin{aligned} x^2-1 &= (x+1)(x-1)\\ x^3-1 &= \underline{\qquad\quad(x-1)(x^2+x+1)} \end{aligned}$$

$\therefore$ L. C. D. $=(x+1)(x-1)(x^2+x+1)=(x+1)(x^3-1)$.

Multiplying the terms of the first fraction by x^2+x+1, and of the second by $x+1$, we have:

$$\frac{x^2+x+1}{(x+1)(x^3-1)}, \quad \frac{x+1}{(x+1)(x^3-1)}.$$

8. $$\begin{aligned} mn-n^2 &= n\qquad\quad(m-n)\\ m^2-n^2 &= \underline{(m+n)(m-n)} \end{aligned}$$

$\therefore$ L. C. D. $=n(m+n)(m-n)=n(m^2-n^2)$.

Multiplying the terms of the first fraction by $n(m^2-n^2)$, of the second by $m+n$, and of the third by n, we have:

$$\frac{mn(m^2-n^2)}{n(m^2-n^2)}, \quad \frac{m^3(m+n)}{n(m^2-n^2)}, \quad \frac{mn^3}{n(m^2-n^2)}.$$

9. The L. C. M. of $a-b$, $a+b$, and a^2+b^2 is $(a-b)(a+b)(a^2+b^2)$, or a^4-b^4. Multiplying the terms of the first fraction by $(a+b)(a^2+b^2)$, of the second by $(a-b)(a^2+b^2)$, and of the third by $(a-b)(a+b)$, we have:

$$\frac{2(a+b)(a^2+b^2)}{a^4-b^4}, \quad \frac{3(a-b)(a^2+b^2)}{a^4-b^4}, \quad \frac{4(a-b)(a+b)}{a^4-b^4};$$

or, $$\frac{2(a^3+a^2b+ab^2+b^3)}{a^4-b^4}, \quad \frac{3(a^3-a^2b+ab^2-b^3)}{a^4-b^4}, \quad \frac{4(a^2-b^2)}{a^4-b^4}.$$

10. The L. C. M. of $1-x$, $(1-x)^2$, and $(1-x)^3$ is $(1-x)^3$. Multiplying the terms of the first fraction by $(1-x)^2$, and of the second by $1-x$, we have:

$$\frac{ay(1-x)^2}{(1-x)^3},\quad \frac{ax^2(1-x)}{(1-x)^3},\quad \frac{xy^3}{(1-x)^3}.$$

11. $$\begin{aligned} am-bm+an-bn &= (a-b)(m+n) \\ 2a^2-2ab &= \underline{2a\,(a-b)} \\ \therefore \text{L. C. D.} &= 2a\,(a-b)(m+n). \end{aligned}$$

Multiplying the terms of the first fraction by $2a$, and of the second by $m+n$, we have:

$$\frac{2a^2b}{2a\,(a-b)(m+n)},\quad \frac{m^2-n^2}{2a\,(a-b)(m+n)}.$$

12. $$\begin{aligned} x^2-3x+2 &= (x-1)(x-2) \\ x^2-5x+6 &= (x-2)(x-3) \\ x^2-4x+3 &= \underline{(x-1)(x-3)} \\ \therefore \text{L. C. D.} &= (x-1)(x-2)(x-3). \end{aligned}$$

Multiplying the terms of the first fraction by $x-3$, of the second by $x-1$, and of the third by $x-2$, we have:

$$\frac{x^2-9}{(x-1)(x-2)(x-3)},\quad \frac{x^2-1}{(x-1)(x-2)(x-3)},\quad \frac{x^2-4}{(x-1)(x-2)(x-3)}.$$

Art. 158.—Pages 92 to 97.

3. $$\frac{2x-5}{12}+\frac{3x+11}{18}=\frac{6x-15}{36}+\frac{6x+22}{36}=\frac{12x+7}{36}.$$

4. $$\frac{3}{5ab^2}-\frac{1}{2a^2b}=\frac{6a}{10a^2b^2}-\frac{5b}{10a^2b^2}=\frac{6a-5b}{10a^2b^2}.$$

5. $$\frac{2a+3}{6}-\frac{3a+5}{8}=\frac{8a+12}{24}-\frac{9a+15}{24}=\frac{8a+12-(9a+15)}{24}$$
$$=\frac{8a+12-9a-15}{24}=\frac{-a-3}{24}=-\frac{a+3}{24}.$$

6. $\frac{m-2}{2mn}-\frac{2-3mn^2}{3m^2n^3}=\frac{3m^2n^2-6mn^2}{6m^2n^3}-\frac{4-6mn^2}{6m^2n^3}$

$=\frac{3m^2n^2-6mn^2-(4-6mn^2)}{6m^2n^3}$

$=\frac{3m^2n^2-6mn^2-4+6mn^2}{6m^2n^3}=\frac{3m^2n^2-4}{6m^2n^3}.$

7. $\frac{b-4a}{24a}+\frac{a+5b}{30b}=\frac{5b^2-20ab}{120ab}+\frac{4a^2+20ab}{120ab}=\frac{4a^2+5b^2}{120ab}.$

8. $\frac{a-b}{4}+\frac{2a+b}{6}-\frac{3a-b}{8}=\frac{6a-6b}{24}+\frac{8a+4b}{24}-\frac{9a-3b}{24}$

$=\frac{6a-6b+8a+4b-(9a-3b)}{24}$

$=\frac{6a-6b+8a+4b-9a+3b}{24}=\frac{5a+b}{24}.$

9. $\frac{a^2+1}{3a^2}-\frac{6a^3+1}{12a^3}+\frac{b-2}{6b}=\frac{4a^3b+4ab}{12a^3b}-\frac{6a^3b+b}{12a^3b}+\frac{2a^3b-4a^3}{12a^3b}$

$=\frac{4a^3b+4ab-(6a^3b+b)+2a^3b-4a^3}{12a^3b}$

$=\frac{4a^3b+4ab-6a^3b-b+2a^3b-4a^3}{12a^3b}=\frac{4ab-b-4a^3}{12a^3b}.$

10. $\frac{2x-1}{12}+\frac{2x+3}{15}-\frac{6x+1}{20}=\frac{10x-5}{60}+\frac{8x+12}{60}-\frac{18x+3}{60}$

$=\frac{10x-5+8x+12-(18x+3)}{60}$

$=\frac{10x-5+8x+12-18x-3}{60}=\frac{4}{60}=\frac{1}{15}.$

11. $\frac{m+2}{7}-\frac{m+2}{14}-\frac{m+3}{21}=\frac{6m+12}{42}-\frac{3m+6}{42}-\frac{2m+6}{42}$

$=\frac{6m+12-(3m+6)-(2m+6)}{42}$

$=\frac{6m+12-3m-6-2m-6}{42}=\frac{m}{42}.$

12. $\frac{2}{3}-\frac{2x-1}{6x}-\frac{3x^2+1}{9x^2}=\frac{12x^2}{18x^2}-\frac{6x^2-3x}{18x^2}-\frac{6x^2+2}{18x^2}$

$$=\frac{12x^2-(6x^2-3x)-(6x^2+2)}{18x^2}$$

$$=\frac{12x^2-6x^2+3x-6x^2-2}{18x^2}=\frac{3x-2}{18x^2}.$$

13. $\frac{x-2}{2}+\frac{3x+1}{3}-\frac{6x-5}{4}-\frac{3}{5}=\frac{30x-60}{60}+\frac{60x+20}{60}-\frac{90x-75}{60}-\frac{36}{60}$

$$=\frac{30x-60+60x+20-(90x-75)-36}{60}$$

$$=\frac{30x-60+60x+20-90x+75-36}{60}=-\frac{1}{60}.$$

14. $\frac{3a+1}{12a}-\frac{2b-1}{8b}+\frac{4c-1}{16c}-\frac{6d+1}{24d}$

$$=\frac{12abcd+4bcd}{48abcd}-\frac{12abcd-6acd}{48abcd}+\frac{12abcd-3abd}{48abcd}-\frac{12abcd+2abc}{48abcd}$$

$$=\frac{12abcd+4bcd-(12abcd-6acd)+12abcd-3abd-(12abcd+2abc)}{48abcd}$$

$$=\frac{12abcd+4bcd-12abcd+6acd+12abcd-3abd-12abcd-2abc}{48abcd}$$

$$=\frac{4bcd+6acd-3abd-2abc}{48abcd}.$$

17. $\frac{1}{1+x}+\frac{1}{1-x}=\frac{1-x}{1-x^2}+\frac{1+x}{1-x^2}=\frac{2}{1-x^2}.$

18. $\frac{1}{x+2}+\frac{1}{3-x}=\frac{3-x}{(x+2)(3-x)}+\frac{x+2}{(x+2)(3-x)}=\frac{5}{6+x-x^2}.$

19. $\frac{1}{x+7}-\frac{1}{x+8}=\frac{x+8}{(x+7)(x+8)}-\frac{x+7}{(x+7)(x+8)}=\frac{x+8-(x+7)}{x^2+15x+56}$

$$=\frac{x+8-x-7}{x^2+15x+56}=\frac{1}{x^2+15x+56}.$$

20. $\frac{a}{a-b}-\frac{b}{a+b}=\frac{a(a+b)}{a^2-b^2}-\frac{b(a-b)}{a^2-b^2}=\frac{a(a+b)-b(a-b)}{a^2-b^2}$

$$=\frac{a^2+ab-ab+b^2}{a^2-b^2}=\frac{a^2+b^2}{a^2-b^2}.$$

21. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}=\dfrac{(a+b)^2}{a^2-b^2}-\dfrac{(a-b)^2}{a^2-b^2}=\dfrac{(a+b)^2-(a-b)^2}{a^2-b^2}$

$$=\frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2-b^2}=\frac{4ab}{a^2-b^2}.$$

22. $\dfrac{x+y}{y}-\dfrac{2xy+x^2}{y(x+y)}=\dfrac{(x+y)^2}{y(x+y)}-\dfrac{2xy+x^2}{y(x+y)}=\dfrac{(x+y)^2-(2xy+x^2)}{y(x+y)}$

$$=\frac{x^2+2xy+y^2-2xy-x^2}{y(x+y)}=\frac{y^2}{y(x+y)}=\frac{y}{x+y}.$$

23. $\dfrac{1+x}{1-x}-\dfrac{1-x}{1+x}=\dfrac{(1+x)^2}{1-x^2}-\dfrac{(1-x)^2}{1-x^2}=\dfrac{(1+x)^2-(1-x)^2}{1-x^2}$

$$=\frac{1+2x+x^2-1+2x-x^2}{1-x^2}=\frac{4x}{1-x^2}.$$

24. $\dfrac{m+n}{(m-n)^2}+\dfrac{2m}{m^2-n^2}=\dfrac{(m+n)^2}{(m+n)(m-n)^2}+\dfrac{2m(m-n)}{(m+n)(m-n)^2}$

$$=\frac{m^2+2mn+n^2+2m^2-2mn}{(m+n)(m-n)^2}=\frac{3m^2+n^2}{(m+n)(m-n)^2}.$$

25. $\dfrac{1}{a^2-4a+4}-\dfrac{1}{a^2+a-6}=\dfrac{1}{(a-2)^2}-\dfrac{1}{(a+3)(a-2)}$

$$=\frac{a+3}{(a+3)(a-2)^2}-\frac{a-2}{(a+3)(a-2)^2}=\frac{a+3-(a-2)}{(a+3)(a-2)^2}$$

$$=\frac{a+3-a+2}{(a+3)(a-2)^2}=\frac{5}{(a+3)(a-2)^2}.$$

26. $\dfrac{x}{x-y}+\dfrac{3x}{x+y}-\dfrac{2xy}{x^2-y^2}=\dfrac{x(x+y)}{x^2-y^2}+\dfrac{3x(x-y)}{x^2-y^2}-\dfrac{2xy}{x^2-y^2}$

$$=\frac{x(x+y)+3x(x-y)-2xy}{x^2-y^2}=\frac{x^2+xy+3x^2-3xy-2xy}{x^2-y^2}$$

$$=\frac{4x^2-4xy}{x^2-y^2}=\frac{4x(x-y)}{(x+y)(x-y)}=\frac{4x}{x+y}.$$

27. $\dfrac{a}{a+b}+\dfrac{b}{a-b}+\dfrac{2ab}{a^2-b^2}=\dfrac{a(a-b)}{a^2-b^2}+\dfrac{b(a+b)}{a^2-b^2}+\dfrac{2ab}{a^2-b^2}$

$$=\frac{a(a-b)+b(a+b)+2ab}{a^2-b^2}=\frac{a^2-ab+ab+b^2+2ab}{a^2-b^2}$$

$$=\frac{a^2+2ab+b^2}{a^2-b^2}=\frac{(a+b)^2}{(a+b)(a-b)}=\frac{a+b}{a-b}.$$

28. $$\frac{m}{mn-n^2}-\frac{1}{m-n}-\frac{1}{n}=\frac{m}{n(m-n)}-\frac{n}{n(m-n)}-\frac{m-n}{n(m-n)}$$

$$=\frac{m-n-(m-n)}{n(m-n)}=\frac{m-n-m+n}{n(m-n)}=\frac{0}{n(m-n)}=0.$$

29. $$\frac{x+y}{x-y}+\frac{x-y}{x+y}+2=\frac{(x+y)^2}{x^2-y^2}+\frac{(x-y)^2}{x^2-y^2}+\frac{2(x^2-y^2)}{x^2-y^2}$$

$$=\frac{(x+y)^2+(x-y)^2+2(x^2-y^2)}{x^2-y^2}$$

$$=\frac{x^2+2xy+y^2+x^2-2xy+y^2+2x^2-2y^2}{x^2-y^2}=\frac{4x^2}{x^2-y^2}.$$

30. $$\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}=\frac{2(4x^2-1)}{x(4x^2-1)}-\frac{3x(2x+1)}{x(4x^2-1)}-\frac{x(2x-3)}{x(4x^2-1)}$$

$$=\frac{2(4x^2-1)-3x(2x+1)-x(2x-3)}{x(4x^2-1)}$$

$$=\frac{8x^2-2-6x^2-3x-2x^2+3x}{x(4x^2-1)}=-\frac{2}{x(4x^2-1)}.$$

31. $$\frac{1}{a+b}+\frac{1}{a-b}-\frac{2a}{a^2+b^2}$$

$$=\frac{(a-b)(a^2+b^2)}{a^4-b^4}+\frac{(a+b)(a^2+b^2)}{a^4-b^4}-\frac{2a(a^2-b^2)}{a^4-b^4}$$

$$=\frac{(a-b)(a^2+b^2)+(a+b)(a^2+b^2)-2a(a^2-b^2)}{a^4-b^4}$$

$$=\frac{a^3-a^2b+ab^2-b^3+a^3+a^2b+ab^2+b^3-2a^3+2ab^2}{a^4-b^4}=\frac{4ab^2}{a^4-b^4}.$$

32. $$\frac{1}{1-x}-\frac{x}{(1-x)^2}-\frac{x^2-4x}{(1-x)^3}=\frac{(1-x)^2}{(1-x)^3}-\frac{x(1-x)}{(1-x)^3}-\frac{x^2-4x}{(1-x)^3}$$

$$=\frac{(1-x)^2-x(1-x)-(x^2-4x)}{(1-x)^3}$$

$$=\frac{1-2x+x^2-x+x^2-x^2+4x}{(1-x)^3}=\frac{1+x+x^2}{(1-x)^3}.$$

33. $$\frac{1}{ab-cd}-\frac{1}{ab+cd}-\frac{2cd}{a^2b^2-c^2d^2}=\frac{ab+cd}{a^2b^2-c^2d^2}-\frac{ab-cd}{a^2b^2-c^2d^2}-\frac{2cd}{a^2b^2-c^2d^2}$$
$$=\frac{ab+cd-(ab-cd)-2cd}{a^2b^2-c^2d^2}=\frac{ab+cd-ab+cd-2cd}{a^2b^2-c^2d^2}=0.$$

34. $$\frac{x-3}{x-2}-\frac{x+1}{x+5}+\frac{x+13}{x^2+3x-10}$$
$$=\frac{(x-3)(x+5)}{(x-2)(x+5)}-\frac{(x+1)(x-2)}{(x-2)(x+5)}+\frac{x+13}{(x-2)(x+5)}$$
$$=\frac{(x-3)(x+5)-(x+1)(x-2)+x+13}{x^2+3x-10}$$
$$=\frac{x^2+2x-15-(x^2-x-2)+x+13}{x^2+3x-10}$$
$$=\frac{x^2+2x-15-x^2+x+2+x+13}{x^2+3x-10}=\frac{4x}{x^2+3x-10}.$$

35 $$\frac{x-a}{x-b}+\frac{x-b}{x-a}-\frac{(a-b)^2}{(x-a)(x-b)}$$
$$=\frac{(x-a)^2}{(x-a)(x-b)}+\frac{(x-b)^2}{(x-a)(x-b)}-\frac{(a-b)^2}{(x-a)(x-b)}$$
$$=\frac{(x-a)^2+(x-b)^2-(a-b)^2}{(x-a)(x-b)}$$
$$=\frac{x^2-2ax+a^2+x^2-2bx+b^2-a^2+2ab-b^2}{(x-a)(x-b)}$$
$$=\frac{2x^2-2ax-2bx+2ab}{(x-a)(x-b)}=\frac{2(x-a)(x-b)}{(x-a)(x-b)}=2.$$

36. $$\frac{1}{x(x+1)}-\frac{1}{x(x-1)}+\frac{x}{x^2-1}=\frac{x-1}{x(x^2-1)}-\frac{x+1}{x(x^2-1)}+\frac{x^2}{x(x^2-1)}$$
$$=\frac{x-1-(x+1)+x^2}{x(x^2-1)}=\frac{x-1-x-1+x^2}{x(x^2-1)}=\frac{x^2-2}{x(x^2-1)}.$$

37. $$\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}$$
$$=\frac{a^2-b^2}{(a-b)(b-c)(c-a)}+\frac{b^2-c^2}{(a-b)(b-c)(c-a)}+\frac{c^2-a^2}{(a-b)(b-c)(c-a)}$$
$$=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(a-b)(b-c)(c-a)}=0.$$

38. $\dfrac{1}{x-1}-\dfrac{x}{x^2-1}+\dfrac{3}{x^3-1}$

$$=\frac{(x+1)(x^2+x+1)}{(x+1)(x^3-1)}-\frac{x(x^2+x+1)}{(x+1)(x^3-1)}+\frac{3(x+1)}{(x+1)(x^3-1)}$$

$$=\frac{(x+1)(x^2+x+1)-x(x^2+x+1)+3(x+1)}{(x+1)(x^3-1)}$$

$$=\frac{x^3+x^2+x+x^2+x+1-x^3-x^2-x+3x+3}{(x+1)(x^3-1)}$$

$$=\frac{x^2+4x+4}{(x+1)(x^3-1)}=\frac{(x+2)^2}{(x+1)(x^3-1)}.$$

39. $\dfrac{2x-6}{x^2+3x+2}-\dfrac{x+2}{x^2-2x-3}-\dfrac{x+1}{x^2-x-6}$

$$=\frac{2(x-3)}{(x+1)(x+2)}-\frac{x+2}{(x+1)(x-3)}-\frac{x+1}{(x+2)(x-3)}$$

$$=\frac{2(x-3)^2}{(x+1)(x+2)(x-3)}-\frac{(x+2)^2}{(x+1)(x+2)(x-3)}-\frac{(x+1)^2}{(x+1)(x+2)(x-3)}$$

$$=\frac{2(x-3)^2-(x+2)^2-(x+1)^2}{(x+1)(x+2)(x-3)}$$

$$=\frac{2x^2-12x+18-x^2-4x-4-x^2-2x-1}{(x+1)(x+2)(x-3)}=\frac{13-18x}{(x+1)(x+2)(x-3)}.$$

40. $\dfrac{x-y}{(x+z)(y+z)}+\dfrac{y-z}{(x+y)(x+z)}-\dfrac{z-x}{(x+y)(y+z)}$

$$=\frac{x^2-y^2}{(x+y)(y+z)(z+x)}+\frac{y^2-z^2}{(x+y)(y+z)(z+x)}-\frac{z^2-x^2}{(x+y)(y+z)(z+x)}$$

$$=\frac{x^2-y^2+y^2-z^2-(z^2-x^2)}{(x+y)(y+z)(z+x)}=\frac{x^2-y^2+y^2-z^2-z^2+x^2}{(x+y)(y+z)(z+x)}$$

$$=\frac{2x^2-2z^2}{(x+y)(y+z)(z+x)}=\frac{2(x+z)(x-z)}{(x+y)(y+z)(z+x)}=\frac{2(x-z)}{(x+y)(y+z)}.$$

43. $\dfrac{4}{a^2-ab}+\dfrac{3}{b^2-ab}=\dfrac{4}{a^2-ab}-\dfrac{3}{ab-b^2}=\dfrac{4}{a(a-b)}-\dfrac{3}{b(a-b)}$

$$=\frac{4b}{ab(a-b)}-\frac{3a}{ab(a-b)}=\frac{4b-3a}{ab(a-b)}.$$

44. $$\frac{5a+1}{3a-3}-\frac{3a-1}{2-2a}=\frac{5a+1}{3a-3}+\frac{3a-1}{2a-2}=\frac{5a+1}{3(a-1)}+\frac{3a-1}{2(a-1)}$$

$$=\frac{10a+2}{6(a-1)}+\frac{9a-3}{6(a-1)}=\frac{19a-1}{6(a-1)}.$$

45. $$\frac{1}{3x-x^2}+\frac{1}{x^2-9}=\frac{1}{3x-x^2}-\frac{1}{9-x^2}=\frac{1}{x(3-x)}-\frac{1}{(3+x)(3-x)}$$

$$=\frac{3+x}{x(9-x^2)}-\frac{x}{x(9-x^2)}=\frac{3+x-x}{x(9-x^2)}=\frac{3}{x(9-x^2)}=\frac{3}{9x-x^3}.$$

46. $$\frac{1}{m^2-mn}-\frac{1}{n^2-m^2}=\frac{1}{m^2-mn}+\frac{1}{m^2-n^2}=\frac{1}{m(m-n)}+\frac{1}{m^2-n^2}$$

$$=\frac{m+n}{m(m^2-n^2)}+\frac{m}{m(m^2-n^2)}=\frac{2m+n}{m(m^2-n^2)}.$$

47. $$\frac{1}{(a-2)(x+2)}+\frac{1}{(2-a)(x+a)}=\frac{1}{(a-2)(x+2)}-\frac{1}{(a-2)(x+a)}$$

$$=\frac{x+a}{(a-2)(x+2)(x+a)}-\frac{x+2}{(a-2)(x+2)(x+a)}$$

$$=\frac{x+a-x-2}{(a-2)(x+2)(x+a)}=\frac{a-2}{(a-2)(x+2)(x+a)}=\frac{1}{(x+2)(x+a)}.$$

48. $$\frac{a}{a+b}+\frac{a}{b-a}+\frac{2a^2}{a^2-b^2}=\frac{a}{a+b}-\frac{a}{a-b}+\frac{2a^2}{a^2-b^2}$$

$$=\frac{a(a-b)}{a^2-b^2}-\frac{a(a+b)}{a^2-b^2}+\frac{2a^2}{a^2-b^2}=\frac{a(a-b)-a(a+b)+2a^2}{a^2-b^2}$$

$$=\frac{a^2-ab-a^2-ab+2a^2}{a^2-b^2}=\frac{2a^2-2ab}{a^2-b^2}=\frac{2a(a-b)}{(a+b)(a-b)}=\frac{2a}{a+b}.$$

49. $$\frac{x}{1+x}-\frac{x}{1-x}-\frac{x^2}{x^2-1}=\frac{x}{x+1}+\frac{x}{x-1}-\frac{x^2}{x^2-1}$$

$$=\frac{x(x-1)}{x^2-1}+\frac{x(x+1)}{x^2-1}-\frac{x^2}{x^2-1}=\frac{x(x-1)+x(x+1)-x^2}{x^2-1}$$

$$=\frac{x^2-x+x^2+x-x^2}{x^2-1}=\frac{x^2}{x^2-1}.$$

50. $\dfrac{3}{2-x}+\dfrac{5}{x-3}+\dfrac{1}{x^2-5x+6}=-\dfrac{3}{x-2}+\dfrac{5}{x-3}+\dfrac{1}{(x-2)(x-3)}$

$$=\frac{-3(x-3)}{(x-2)(x-3)}+\frac{5(x-2)}{(x-2)(x-3)}+\frac{1}{(x-2)(x-3)}$$

$$=\frac{-3(x-3)+5(x-2)+1}{x^2-5x+6}$$

$$=\frac{-3x+9+5x-10+1}{x^2-5x+6}=\frac{2x}{x^2-5x+6}.$$

51. $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(b-a)(a-c)}-\dfrac{1}{(c-a)(c-b)}$

$$=\frac{1}{(a-b)(b-c)}+\frac{1}{(a-b)(c-a)}+\frac{1}{(c-a)(b-c)}$$

$$=\frac{c-a+b-c+a-b}{(a-b)(b-c)(c-a)}=0.$$

52. $\dfrac{2}{(x-2)(x-3)}-\dfrac{3}{(3-x)(4-x)}-\dfrac{1}{(x-4)(2-x)}$

$$=\frac{2}{(x-2)(x-3)}-\frac{3}{(x-3)(x-4)}+\frac{1}{(x-4)(x-2)}$$

$$=\frac{2(x-4)-3(x-2)+x-3}{(x-2)(x-3)(x-4)}=\frac{2x-8-3x+6+x-3}{(x-2)(x-3)(x-4)}$$

$$=-\frac{5}{(x-2)(x-3)(x-4)}.$$

Art. 159.—Pages 98, 99.

3. $\dfrac{5a^2bc}{12mn^2}\times 3mn=\dfrac{5a^2bc}{4n}.$ 4. $\dfrac{3abx^2}{5ay^2}\times\dfrac{5xy^2}{3abx^3}=\dfrac{3\cdot 5\cdot abx^3y^2}{5\cdot 3\cdot a^2bx^3y^2}=\dfrac{1}{a}.$

5. $\dfrac{2a}{3b}\times\dfrac{6c}{5a}\times\dfrac{5b}{8c}=\dfrac{2\cdot 6\cdot 5\cdot abc}{3\cdot 5\cdot 8\cdot abc}=\dfrac{1}{2}.$

6. $\dfrac{8x^2}{9y^3}\times\dfrac{15y^2}{16z^3}\times\dfrac{3z^4}{10x^3}=\dfrac{8\cdot 15\cdot 3\cdot x^2y^2z^4}{9\cdot 16\cdot 10\cdot x^3y^3z^3}=\dfrac{z}{4xy}.$

7. $\dfrac{3ab^2}{4cd}\times\dfrac{3ac^2}{2bd}\times\dfrac{8ad^2}{9bc}=\dfrac{3\cdot 3\cdot 8\cdot a^3b^2c^2d^2}{4\cdot 2\cdot 9\cdot b^2c^2d^2}=a^3.$

8. $\dfrac{3m^3}{4x^2}\times\dfrac{2n^4}{21m^2}\times\dfrac{7x^2}{5mn^3}=\dfrac{3\cdot2\cdot7\cdot m^3n^4x^2}{4\cdot21\cdot5\cdot m^3n^3x^2}=\dfrac{n}{10}.$

9. $\dfrac{3x^2-x}{5}\times\dfrac{10}{2x^2-4x}=\dfrac{x(3x-1)}{5}\times\dfrac{10}{2x(x-2)}=\dfrac{3x-1}{x-2}.$

10. $\dfrac{x^2-16}{x^2+5x}\times\dfrac{x^2-25}{x^2-4x}=\dfrac{(x+4)(x-4)}{x(x+5)}\times\dfrac{(x+5)(x-5)}{x(x-4)}$

$=\dfrac{(x+4)(x-5)}{x^2}=\dfrac{x^2-x-20}{x^2}.$

11. $\dfrac{a-b}{a^2+2ab}\times\dfrac{a^2-4b^2}{a^2-ab}=\dfrac{a-b}{a(a+2b)}\times\dfrac{(a+2b)(a-2b)}{a(a-b)}=\dfrac{a-2b}{a^2}.$

12. $\dfrac{a^2-2ab+b^2}{a+b}\times\dfrac{b}{ax-bx}=\dfrac{(a-b)^2}{a+b}\times\dfrac{b}{x(a-b)}=\dfrac{b(a-b)}{x(a+b)}=\dfrac{ab-b^2}{ax+bx}.$

13. $\dfrac{xy+y^2}{x^3-y^3}\times\dfrac{x^3+x^2y+xy^2}{(x+y)^2}=\dfrac{y(x+y)}{(x-y)(x^2+xy+y^2)}\times\dfrac{x(x^2+xy+y^2)}{(x+y)^2}$

$=\dfrac{xy}{(x+y)(x-y)}=\dfrac{xy}{x^2-y^2}.$

14. $\dfrac{a^3-a^2+a}{x^2+2x+4}\times\dfrac{x^3-8}{a^3+1}=\dfrac{a(a^2-a+1)}{x^2+2x+4}\times\dfrac{(x-2)(x^2+2x+4)}{(a+1)(a^2-a+1)}$

$=\dfrac{a(x-2)}{a+1}=\dfrac{ax-2a}{a+1}.$

15. $\left(1+\dfrac{4}{x}-\dfrac{5}{x^2}\right)\times\dfrac{3x}{x^2+x-2}=\dfrac{x^2+4x-5}{x^2}\times\dfrac{3x}{x^2+x-2}$

$=\dfrac{(x+5)(x-1)}{x^2}\times\dfrac{3x}{(x+2)(x-1)}=\dfrac{3(x+5)}{x(x+2)}=\dfrac{3x+15}{x^2+2x}.$

16. $\dfrac{1-x^2}{1-y}\times\dfrac{1-y^2}{x+x^2}\times\dfrac{1}{1-x}=\dfrac{(1+x)(1-x)}{1-y}\times\dfrac{(1+y)(1-y)}{x(1+x)}\times\dfrac{1}{1-x}=\dfrac{1+y}{x}.$

17. $\dfrac{x^2+5xy+6y^2}{x^2-4xy-21y^2}\times\dfrac{x^2-7xy}{x^2-4y^2}$

$=\dfrac{(x+2y)(x+3y)}{(x-7y)(x+3y)}\times\dfrac{x(x-7y)}{(x+2y)(x-2y)}=\dfrac{x}{x-2y}.$

18. $\dfrac{x^2y-4y}{(x-y)^2-z^2}\times\dfrac{x^2-xy-xz}{xy+2y}=\dfrac{y(x+2)(x-2)}{(x-y+z)(x-y-z)}\times\dfrac{x(x-y-z)}{y(x+2)}$

$=\dfrac{x(x-2)}{x-y+z}=\dfrac{x^2-2x}{x-y+z}.$

19. $\dfrac{x^3-y^3}{x^2-xy+y^2}\times\dfrac{x^3+y^3}{x^2+xy+y^2}\times\left(1+\dfrac{y}{x-y}\right)$

$=\dfrac{(x-y)(x^2+xy+y^2)}{x^2-xy+y^2}\times\dfrac{(x+y)(x^2-xy+y^2)}{x^2+xy+y^2}\times\dfrac{x}{x-y}$

$=x(x+y)=x^2+xy.$

20. $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}\times\dfrac{a^2-(b+c)^2}{(a-c)^2-b^2}$

$=\dfrac{(a+b-c)(a-b+c)}{(a+b+c)(a-b+c)}\times\dfrac{(a+b+c)(a-b-c)}{(a+b-c)(a-b-c)}=1.$

Art. 160.—Pages 100, 101.

3. $\dfrac{7a^3b}{5m^2n^3}\div 14ab^4=\dfrac{7a^3b}{5\cdot 14\cdot ab^4m^2n^3}=\dfrac{a^2}{10b^3m^2n^3}.$

4. $\dfrac{18mx^3}{25ny^2}\div\dfrac{6m^2x^4}{5n^2y^5}=\dfrac{18mx^3}{25ny^2}\times\dfrac{5n^2y^5}{6m^2x^4}=\dfrac{18\cdot 5\cdot mn^2x^3y^5}{25\cdot 6\cdot m^2nx^4y^2}=\dfrac{3ny^3}{5mx}.$

5. $\dfrac{1}{a^2+a-12}\div\dfrac{1}{a^2+3a-18}=\dfrac{1}{(a+4)(a-3)}\times(a+6)(a-3)=\dfrac{a+6}{a+4}.$

6. $\left(\dfrac{1}{4}-\dfrac{4}{x^2}\right)\div\left(\dfrac{x^2}{12}+\dfrac{x}{3}\right)=\dfrac{x^2-16}{4x^2}\div\dfrac{x^2+4x}{12}$

$=\dfrac{(x+4)(x-4)}{4x^2}\times\dfrac{12}{x(x+4)}=\dfrac{3(x-4)}{x^3}=\dfrac{3x-12}{x^3}.$

7. $\dfrac{x^3-25x}{x^2+x-6}\div\dfrac{x^2-5x}{x^2-x-12}=\dfrac{x(x+5)(x-5)}{(x+3)(x-2)}\times\dfrac{(x-4)(x+3)}{x(x-5)}$

$=\dfrac{(x+5)(x-4)}{x-2}=\dfrac{x^2+x-20}{x-2}.$

8. $\dfrac{ab-b^2}{a^2+2ab+b^2}\div\dfrac{b^2}{a^2-b^2}=\dfrac{b(a-b)}{(a+b)^2}\times\dfrac{(a+b)(a-b)}{b^2}=\dfrac{(a-b)^2}{b(a+b)}=\dfrac{(a-b)^2}{ab+b^2}.$

9. $\dfrac{m^3+n^3}{m^2-2mn+n^2}\div\dfrac{m^2+mn}{m-n}=\dfrac{(m+n)(m^2-mn+n^2)}{(m-n)^2}\times\dfrac{m-n}{m(m+n)}$

$=\dfrac{m^2-mn+n^2}{m(m-n)}=\dfrac{m^2-mn+n^2}{m^2-mn}.$

10. $\dfrac{a+1}{a^2-3a}\div\dfrac{a^2-a-2}{a^2-a-6}=\dfrac{a+1}{a(a-3)}\times\dfrac{(a-3)(a+2)}{(a-2)(a+1)}=\dfrac{a+2}{a(a-2)}=\dfrac{a+2}{a^2-2a}.$

11. $\left(9+\dfrac{5y^2}{x^2-y^2}\right)\div\left(3+\dfrac{5y}{x-y}\right)=\dfrac{9x^2-4y^2}{x^2-y^2}\div\dfrac{3x+2y}{x-y}$

$=\dfrac{(3x+2y)(3x-2y)}{(x+y)(x-y)}\times\dfrac{x-y}{3x+2y}=\dfrac{3x-2y}{x+y}.$

12. $\dfrac{a^4-8ab^3}{a^2-2ab-3b^2}\div\dfrac{a^3+2a^2b+4ab^2}{a-3b}$

$=\dfrac{a(a-2b)(a^2+2ab+4b^2)}{(a+b)(a-3b)}\times\dfrac{a-3b}{a(a^2+2ab+4b^2)}=\dfrac{a-2b}{a+b}.$

13. $\left(\dfrac{2}{3y^2}-\dfrac{2}{xy}+\dfrac{3}{2x^2}\right)\div\left(\dfrac{2}{3y^2}-\dfrac{3}{2x^2}\right)=\dfrac{4x^2-12xy+9y^2}{6x^2y^2}\div\dfrac{4x^2-9y^2}{6x^2y^2}$

$=\dfrac{(2x-3y)^2}{6x^2y^2}\times\dfrac{6x^2y^2}{(2x+3y)(2x-3y)}=\dfrac{2x-3y}{2x+3y}.$

Art. 161.—Pages 102, 103.

4. $\dfrac{x-\frac{1}{x}}{1+\frac{1}{x}}$, multiplying each term by x, $=\dfrac{x^2-1}{x+1}=x-1.$

5. $\dfrac{\frac{1}{b}-\frac{1}{a}}{\frac{a}{b}-\frac{b}{a}}$, multiplying each term by ab, $=\dfrac{a-b}{a^2-b^2}=\dfrac{1}{a+b}.$

6. $\dfrac{x^2+\frac{1}{x}}{1+\frac{1}{x}}$, multiplying each term by x, $=\dfrac{x^3+1}{x+1}=x^2-x+1.$

7. $\dfrac{a-2+\dfrac{1}{a}}{1-\dfrac{1}{a}}$, multiplying each term by a,

$$=\frac{a^2-2a+1}{a-1}=\frac{(a-1)^2}{a-1}=a-1.$$

8. $\dfrac{\dfrac{x^2+4y^2}{y}-4x}{\dfrac{1}{y}-\dfrac{2}{x}}$, multiplying each term by xy,

$$=\frac{x(x^2+4y^2)-4x^2y}{x-2y}=\frac{x(x^2-4xy+4y^2)}{x-2y}=\frac{x(x-2y)^2}{x-2y}$$

$$=x(x-2y)=x^2-2xy.$$

9. $\dfrac{x-7+\dfrac{12}{x}}{x+3-\dfrac{18}{x}}$, multiplying each term by x,

$$=\frac{x^2-7x+12}{x^2+3x-18}=\frac{(x-3)(x-4)}{(x-3)(x+6)}=\frac{x-4}{x+6}.$$

10. $\dfrac{\dfrac{m^2}{n^3}+\dfrac{1}{m}}{\dfrac{m}{n^2}-\dfrac{m-n}{mn}}$, multiplying each term by mn^3,

$$=\frac{m^3+n^3}{m^2n-n^2(m-n)}=\frac{(m+n)(m^2-mn+n^2)}{n(m^2-mn+n^2)}=\frac{m+n}{n}.$$

11. $\dfrac{x-1-\dfrac{12}{x+3}}{x-5+\dfrac{12}{x+3}}$, multiplying each term by $x+3$,

$$=\frac{(x+3)(x-1)-12}{(x+3)(x-5)+12}=\frac{x^2+2x-3-12}{x^2-2x-15+12}=\frac{x^2+2x-15}{x^2-2x-3}$$

$$=\frac{(x+5)(x-3)}{(x+1)(x-3)}=\frac{x+5}{x+1}.$$

12. $$2-\cfrac{1}{3+\cfrac{1}{\frac{x}{2}-2}}=2-\cfrac{1}{3+\cfrac{2}{x-4}}=2-\frac{x-4}{3x-12+2}=2-\frac{x-4}{3x-10}$$

$$=\frac{6x-20-x+4}{3x-10}=\frac{5x-16}{3x-10}.$$

13. $$\frac{\frac{a}{b}-\frac{b^2}{a^2}}{\frac{a}{b}+1+\frac{b}{a}},$$ multiplying each term by a^2b,

$$=\frac{a^3-b^3}{a^3+a^2b+ab^2}=\frac{(a-b)(a^2+ab+b^2)}{a(a^2+ab+b^2)}=\frac{a-b}{a}.$$

14. $$\frac{\frac{x^2}{y^2}-4+\frac{3y^2}{x^2}}{\frac{x}{y}-\frac{3y}{x}},$$ multiplying each term by x^2y^2,

$$=\frac{x^4-4x^2y^2+3y^4}{x^3y-3xy^3}=\frac{(x^2-y^2)(x^2-3y^2)}{xy(x^2-3y^2)}=\frac{x^2-y^2}{xy}.$$

15. $$\frac{\frac{1}{1-x}-\frac{1}{1+x}}{\frac{1}{1-x}+\frac{1}{1+x}},$$ multiplying each term by $(1+x)(1-x)$,

$$=\frac{1+x-(1-x)}{1+x+1-x}=\frac{1+x-1+x}{1+x+1-x}=\frac{2x}{2}=x.$$

16. $$\frac{1-\frac{2b-2c}{a+b-c}}{1+\frac{2c}{a-b-c}}=\frac{\frac{a+b-c-2b+2c}{a+b-c}}{\frac{a-b-c+2c}{a-b-c}}=\frac{a-b+c}{a+b-c}\times\frac{a-b-c}{a-b+c}=\frac{a-b-c}{a+b-c}.$$

17. $$\frac{1+\frac{2x^2}{1-x^2}}{1-x^2+\frac{4x^2}{1-x^2}},$$ multiplying each term by $1-x^2$,

$$=\frac{1-x^2+2x^2}{(1-x^2)^2+4x^2}=\frac{1+x^2}{1-2x^2+x^4+4x^2}=\frac{1+x^2}{1+2x^2+x^4}$$

$$=\frac{1+x^2}{(1+x^2)^2}=\frac{1}{1+x^2}.$$

18. $\dfrac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}-\dfrac{a-b}{c+d}}$, multiplying each term by $(c+d)(c-d)$,

$$=\frac{(a+b)(c-d)+(a-b)(c+d)}{(a+b)(c+d)-(a-b)(c-d)}=\frac{ac-ad+bc-bd+ac+ad-bc-bd}{ac+ad+bc+bd-ac+ad+bc-bd}$$

$$=\frac{2ac-2bd}{2ad+2bc}=\frac{ac-bd}{ad+bc}.$$

19. $$\frac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}=\frac{1}{x+\dfrac{3-x}{3-x+x+1}}=\frac{1}{x+\dfrac{3-x}{4}}=\frac{4}{4x+3-x}=\frac{4}{3x+3}$$

20. $\dfrac{\dfrac{x+2y}{x+y}+\dfrac{x}{y}}{\dfrac{x+2y}{y}-\dfrac{x}{x+y}}$, multiplying each term by $y(x+y)$,

$$=\frac{y(x+2y)+x(x+y)}{(x+2y)(x+y)-xy}=\frac{xy+2y^2+x^2+xy}{x^2+3xy+2y^2-xy}=\frac{x^2+2xy+2y^2}{x^2+2xy+2y^2}=1$$

21. $\dfrac{x-3a+\dfrac{4a^2}{a+x}}{x-\dfrac{2a^2}{a+x}}$, multiplying each term by $a+x$,

$$=\frac{(x-3a)(a+x)+4a^2}{x(a+x)-2a^2}=\frac{x^2-2ax-3a^2+4a^2}{ax+x^2-2a^2}$$

$$=\frac{x^2-2ax+a^2}{x^2+ax-2a^2}=\frac{(x-a)^2}{(x+2a)(x-a)}=\frac{x-a}{x+2a}.$$

22. $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$, multiplying each term by $(a^2+b^2)(a^2-b^2)$,

$$=\frac{(a^2+b^2)^2-(a^2-b^2)^2}{(a^2+b^2)[(a+b)^2-(a-b)^2]}=\frac{a^4+2a^2b^2+b^4-a^4+2a^2b^2-b^4}{(a^2+b^2)(a^2+2ab+b^2-a^2+2ab-b^2)}$$

$$=\frac{4a^2b^2}{(a^2+b^2)4ab}=\frac{ab}{a^2+b^2}.$$

23. $\dfrac{\dfrac{m-n}{m+n}+\dfrac{m^2+n^2}{m^2-n^2}}{\dfrac{m^4}{m-n}+\dfrac{m^2n+n^4}{(m-n)^2}}$, multiplying each term by $(m+n)(m-n)^2$,

$$=\frac{(m-n)[(m-n)^2+m^2+n^2]}{(m+n)[m^2(m-n)+m^2n+n^3]}=\frac{(m-n)(m^2-2mn+n^2+m^2+n^2)}{(m+n)(m^3-m^2n+m^2n+n^3)}$$

$$=\frac{(m-n)(2m^2-2mn+2n^2)}{(m+n)(m^3+n^3)}$$

$$=\frac{2(m-n)(m^2-mn+n^2)}{(m+n)(m+n)(m^2-mn+n^2)}=\frac{2(m-n)}{(m+n)^2}.$$

Art. 162.—Pages 104, 105.

1. $c+\dfrac{2b}{x}-\dfrac{a+bx+cx^2}{x^2}=\dfrac{cx^2+2bx-(a+bx+cx^2)}{x^2}$

$$=\frac{cx^2+2bx-a-bx-cx^2}{x^2}=\frac{bx-a}{x^2}.$$

2. $\dfrac{m^3+4m^2-5m}{3m^3-75m}=\dfrac{m(m+5)(m-1)}{3m(m+5)(m-5)}=\dfrac{m-1}{3(m-5)}=\dfrac{m-1}{3m-15}.$

3. $\dfrac{x^2(1+x)^3-x^3(1+x)^2}{(1+x)^6}$, dividing each term by $(1+x)^2$,

$$=\frac{x^2(1+x)-x^3}{(1+x)^4}=\frac{x^2+x^3-x^3}{(1+x)^4}=\frac{x^2}{(1+x)^4}.$$

4. $\dfrac{a^2-b^2}{(m+n)^2}\div\dfrac{a^2-ab}{bm+bn}=\dfrac{(a+b)(a-b)}{(m+n)^2}\times\dfrac{b(m+n)}{a(a-b)}=\dfrac{b(a+b)}{a(m+n)}=\dfrac{ab+b^2}{am+an}.$

5. $\dfrac{1+2x^2}{2+2x^2}-\dfrac{2+x}{2+2x}=\dfrac{1+2x^2}{2(1+x^2)}-\dfrac{2+x}{2(1+x)}$

$$=\frac{(1+x)(1+2x^2)-(2+x)(1+x^2)}{2(1+x)(1+x^2)}$$

$$=\frac{1+x+2x^2+2x^3-2-x-2x^2-x^3}{2(1+x)(1+x^2)}=\frac{x^3-1}{2(1+x)(1+x^2)}.$$

6. $\dfrac{10a^2+30ab+20b^2}{5a^3+10a^2b}=\dfrac{10(a+b)(a+2b)}{5a^2(a+2b)}=\dfrac{2(a+b)}{a^2}.$

7. $\left(x+1+\frac{1}{x}\right)\left(x-1+\frac{1}{x}\right)=\left[\left(x+\frac{1}{x}\right)+1\right]\left[\left(x+\frac{1}{x}\right)-1\right]$

$$=\left(x+\frac{1}{x}\right)^2-1=x^2+2+\frac{1}{x^2}-1=x^2+1+\frac{1}{x^2}.$$

8. $\dfrac{1-ax+a(x+a)}{(1-ax)^2+(x+a)^2}=\dfrac{1-ax+ax+a^2}{1-2ax+a^2x^2+x^2+2ax+a^2}$

$$=\frac{1+a^2}{1+a^2+x^2+a^2x^2}=\frac{1+a^2}{(1+a^2)(1+x^2)}=\frac{1}{1+x^2}.$$

9. $\left(\dfrac{a^2}{b^2}-2+\dfrac{b^2}{a^2}\right)\div\left(\dfrac{a}{b}-\dfrac{b}{a}\right)=\dfrac{a^4-2a^2b^2+b^4}{a^2b^2}\div\dfrac{a^2-b^2}{ab}$

$$=\frac{(a^2-b^2)^2}{a^2b^2}\times\frac{ab}{a^2-b^2}=\frac{a^2-b^2}{ab}=\frac{a^2}{ab}-\frac{b^2}{ab}=\frac{a}{b}-\frac{b}{a}.$$

10. $\dfrac{b(b-ax)+a(a+bx)}{(b-ax)^2+(a+bx)^2}=\dfrac{b^2-abx+a^2+abx}{b^2-2abx+a^2x^2+a^2+2abx+b^2x^2}$

$$=\frac{a^2+b^2}{a^2+b^2+a^2x^2+b^2x^2}=\frac{a^2+b^2}{(a^2+b^2)(1+x^2)}=\frac{1}{1+x^2}.$$

11. $\dfrac{ax}{ax+b}-\dfrac{b}{ax-b}+\dfrac{ax(3b-ax)}{a^2x^2-b^2}=\dfrac{ax(ax-b)-b(ax+b)+ax(3b-ax)}{a^2x^2-b^2}$

$$=\frac{a^2x^2-abx-abx-b^2+3abx-a^2x^2}{a^2x^2-b^2}$$

$$=\frac{abx-b^2}{a^2x^2-b^2}=\frac{b(ax-b)}{(ax+b)(ax-b)}=\frac{b}{ax+b}.$$

12. $\dfrac{3x-3x^2}{2+4x+2x^2}\times\dfrac{10x+10x^2}{9-18x+9x^2}=\dfrac{3x(1-x)}{2(1+x)^2}\times\dfrac{10x(1+x)}{9(1-x)^2}$

$$=\frac{5x^2}{3(1+x)(1-x)}=\frac{5x^2}{3-3x^2}.$$

13. $\dfrac{6n^5-48n^2}{9n^5+18n^4+36n^3}=\dfrac{6n^2(n-2)(n^2+2n+4)}{9n^3(n^2+2n+4)}=\dfrac{2(n-2)}{3n}=\dfrac{2n-4}{3n}.$

14. $x^2-2y^2-\dfrac{6y^3-x^2y+xy^2}{x-3y}=\dfrac{(x^2-2y^2)(x-3y)-(6y^3-x^2y+xy^2)}{x-3y}$

$=\dfrac{x^3-3x^2y-2xy^2+6y^3-6y^3+x^2y-xy^2}{x-3y}=\dfrac{x^3-2x^2y-3xy^2}{x-3y}$

$=\dfrac{x(x-3y)(x+y)}{x-3y}=x(x+y)=x^2+xy.$

15. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}-\dfrac{4b^2}{a^2-b^2}=\dfrac{(a+b)^2-(a-b)^2-4b^2}{a^2-b^2}$

$=\dfrac{a^2+2ab+b^2-a^2+2ab-b^2-4b^2}{a^2-b^2}$

$=\dfrac{4ab-4b^2}{a^2-b^2}=\dfrac{4b(a-b)}{(a+b)(a-b)}=\dfrac{4b}{a+b}.$

16.
$$\begin{array}{l}
x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}\\
x-\dfrac{1}{x}\\
\hline
x^4+x^2+1+\dfrac{1}{x^2}\\
\quad -x^2-1-\dfrac{1}{x^2}-\dfrac{1}{x^4}\\
\hline
x^4 \qquad\qquad -\dfrac{1}{x^4}
\end{array}$$

17. $\dfrac{2x-1}{2x^2-2x+1}-\dfrac{2x+1}{2x^2+2x+1}=\dfrac{(2x-1)(2x^2+2x+1)-(2x+1)(2x^2-2x+1)}{(2x^2+2x+1)(2x^2-2x+1)}$

$=\dfrac{4x^3+2x^2-1-(4x^3-2x^2+1)}{[(2x^2+1)+2x][(2x^2+1)-2x]}=\dfrac{4x^3+2x^2-1-4x^3+2x^2-1}{(2x^2+1)^2-4x^2}$

$=\dfrac{4x^2-2}{4x^4+4x^2+1-4x^2}=\dfrac{4x^2-2}{4x^4+1}.$

18. $\dfrac{\frac{1}{a}}{1+\frac{x^2}{a^2}}+\dfrac{1}{2}\left(\dfrac{1}{x-a}-\dfrac{1}{x+a}\right)=\dfrac{\frac{1}{a}}{\frac{a^2+x^2}{a^2}}+\dfrac{1}{2}\times\dfrac{x+a-(x-a)}{x^2-a^2}$

$=\dfrac{1}{a}\times\dfrac{a^2}{a^2+x^2}+\dfrac{1}{2}\times\dfrac{x+a-x+a}{x^2-a^2}=\dfrac{a}{x^2+a^2}+\dfrac{1}{2}\times\dfrac{2a}{x^2-a^2}$

$=\dfrac{a}{x^2+a^2}+\dfrac{a}{x^2-a^2}=\dfrac{a(x^2-a^2)+a(x^2+a^2)}{x^4-a^4}$

$=\dfrac{a(x^2-a^2+x^2+a^2)}{x^4-a^4}=\dfrac{2ax^2}{x^4-a^4}.$

19. $\dfrac{\dfrac{x}{x-y}-\dfrac{y}{x+y}}{\dfrac{x^2}{x^2+y^2}+\dfrac{y^2}{x^2-y^2}}$, multiplying each term by $(x^2+y^2)(x^2-y^2)$,

$$=\frac{(x^2+y^2)[x(x+y)-y(x-y)]}{x^2(x^2-y^2)+y^2(x^2+y^2)}=\frac{(x^2+y^2)(x^2+xy-xy+y^2)}{x^4-x^2y^2+x^2y^2+y^4}$$

$$=\frac{(x^2+y^2)(x^2+y^2)}{x^4+y^4}=\frac{(x^2+y^2)^2}{x^4+y^4}.$$

20.
$$\begin{array}{r|l} x^3-9x^2+26x-24)\,x^3-12x^2+47x-60\,(1 \\ x^3-9x^2+26x-24 \\ \hline -3x^2+21x-36 \end{array}$$

Dividing by -3,
$$\begin{array}{r} x^2-7x+12)\,x^3-9x^2+26x-24\,(x-2 \\ x^3-7x^2+12x \\ \hline -2x^2+14x-24 \\ -2x^2+14x-24 \\ \hline \end{array}$$

$\therefore$ H.C.F. $=x^2-7x+12.$

$$\begin{array}{r} x^2-7x+12)\,x^3-12x^2+47x-60\,(x-5 \\ x^3-7x^2+12x \\ \hline -5x^2+35x-60 \\ -5x^2+35x-60 \\ \hline \end{array}$$

$$\therefore\ \frac{x^3-9x^2+26x-24}{x^3-12x^2+47x-60}=\frac{x-2}{x-5}.$$

21. $a^2-3ab-2b^2-\dfrac{b^2(7a+6b)}{a-3b}=\dfrac{(a-3b)(a^2-3ab-2b^2)-b^2(7a+6b)}{a-3b}$

$$=\frac{a^3-6a^2b+7ab^2+6b^3-7ab^2-6b^3}{a-3b}=\frac{a^3-6a^2b}{a-3b}=\frac{a^2(a-6b)}{a-3b}.$$

22. $\dfrac{2x+y}{x+y}-1-\dfrac{y}{y-x}-\dfrac{x^2}{x^2-y^2}=\dfrac{2x+y}{x+y}-1+\dfrac{y}{x-y}-\dfrac{x^2}{x^2-y^2}$

$$=\frac{(2x+y)(x-y)-(x^2-y^2)+y(x+y)-x^2}{x^2-y^2}$$

$$=\frac{2x^2-xy-y^2-x^2+y^2+xy+y^2-x^2}{x^2-y^2}=\frac{y^2}{x^2-y^2}.$$

23. $$\frac{(x+y+z)^2+(x-y)^2+(y-z)^2+(z-x)^2}{x^2+y^2+z^2}$$
$$=\frac{x^2+y^2+z^2+2xy+2yz+2zx+x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2}{x^2+y^2+z^2}$$
$$=\frac{3x^2+3y^2+3z^2}{x^2+y^2+z^2}=3.$$

24. $$\frac{1}{x-2}-\frac{4}{(x-2)^2}-\frac{8(1-x)}{(x-2)^3}=\frac{(x-2)^2-4(x-2)-8(1-x)}{(x-2)^3}$$
$$=\frac{x^2-4x+4-4x+8-8+8x}{(x-2)^3}=\frac{x^2+4}{(x-2)^3}.$$

25. $$\frac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}=\frac{(a^4-b^4)(a-b)}{(a^3-ab^2)(a-b)}$$
$$=\frac{(a^2+b^2)(a+b)(a-b)(a-b)}{a(a+b)(a-b)(a-b)}=\frac{a^2+b^2}{a}.$$

26. $$\frac{(4x+y)^2-(x-2y)^2}{(3x-4y)^2-(2x+3y)^2}=\frac{(4x+y+x-2y)(4x+y-x+2y)}{(3x-4y+2x+3y)(3x-4y-2x-3y)}$$
$$=\frac{(5x-y)(3x+3y)}{(5x-y)(x-7y)}=\frac{3x+3y}{x-7y}.$$

27. $$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{(a+b+c)^2}{(a+b)(b+c)(c+a)}$$
$$=\frac{(b+c)(c+a)+(c+a)(a+b)+(a+b)(b+c)-(a+b+c)^2}{(a+b)(b+c)(c+a)}$$
$$=\frac{bc+ab+c^2+ac+ac+bc+a^2+ab+ab+ac+b^2+bc-a^2-b^2-c^2-2ab-2bc-2ac}{(a+b)(b+c)(c+a)}$$
$$=\frac{ab+bc+ca}{(a+b)(b+c)(c+a)}.$$

28. $$\frac{2(1-3x)}{(1+x)(1+9x)}-\frac{1-2x}{(1+x)(1+4x)}+\frac{2}{1+4x}$$
$$=\frac{2(1-3x)(1+4x)-(1-2x)(1+9x)+2(1+x)(1+9x)}{(1+x)(1+4x)(1+9x)}$$
$$=\frac{2(1+x-12x^2)-(1+7x-18x^2)+2(1+10x+9x^2)}{(1+x)(1+4x)(1+9x)}$$
$$=\frac{2+2x-24x^2-1-7x+18x^2+2+20x+18x^2}{(1+x)(1+4x)(1+9x)}$$
$$=\frac{3+15x+12x^2}{(1+x)(1+4x)(1+9x)}=\frac{3(1+x)(1+4x)}{(1+x)(1+4x)(1+9x)}=\frac{3}{1+9x}.$$

29. $\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{3x^2}{x^3+1}-\dfrac{3x^2}{x^3-1}$

$$=\frac{(x^5+x^4+x^3+x^2+x+1)-(x^5-x^4+x^3-x^2+x-1)+3x^2(x^3-1)-3x^2(x^3+1)}{x^6-1}$$

$$=\frac{x^5+x^4+x^3+x^2+x+1-x^5+x^4-x^3+x^2-x+1+3x^5-3x^2-3x^5-3x^2}{x^6-1}$$

$$=\frac{2x^4-4x^2+2}{x^6-1}=\frac{2(x^2-1)^2}{(x^2-1)(x^4+x^2+1)}=\frac{2(x^2-1)}{x^4+x^2+1}=\frac{2x^2-2}{x^4+x^2+1}.$$

30. $\left(\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}\right)\div\left(\dfrac{a-b}{a+b}-\dfrac{a^3-b^3}{a^3+b^3}\right)$

$$=\frac{(a+b)^2+a^2+b^2}{a^2-b^2}\div\frac{(a-b)(a^2-ab+b^2)-(a^3-b^3)}{a^3+b^3}$$

$$=\frac{a^2+2ab+b^2+a^2+b^2}{a^2-b^2}\div\frac{a^3-2a^2b+2ab^2-b^3-a^3+b^3}{a^3+b^3}$$

$$=\frac{2a^2+2ab+2b^2}{a^2-b^2}\div-\frac{2a^2b-2ab^2}{a^3+b^3}$$

$$=-\frac{2(a^2+ab+b^2)}{(a+b)(a-b)}\times\frac{(a+b)(a^2-ab+b^2)}{2ab(a-b)}$$

$$=-\frac{(a^2+ab+b^2)(a^2-ab+b^2)}{ab(a-b)^2}=-\frac{[(a^2+b^2)+ab][(a^2+b^2)-ab]}{ab(a-b)^2}$$

$$=-\frac{(a^2+b^2)^2-a^2b^2}{ab(a-b)^2}=-\frac{a^4+2a^2b^2+b^4-a^2b^2}{ab(a-b)^2}=-\frac{a^4+a^2b^2+b^4}{ab(a-b)^2}.$$

CHAPTER XII.

Art. 174.—Pages 109, 110.

3.
$$\begin{aligned} 8x &= 5x + 42 \\ 8x - 5x &= 42 \\ 3x &= 42 \\ x &= 14 \end{aligned}$$

4.
$$\begin{aligned} 7x + 5 &= -30 \\ 7x &= -30 - 5 \\ 7x &= -35 \\ x &= -5 \end{aligned}$$

5.
$$\begin{aligned} 7x + 5 &= x + 23 \\ 7x - x &= 23 - 5 \\ 6x &= 18 \\ x &= 3 \end{aligned}$$

6.
$$\begin{aligned} 9x + 7 &= 3x - 11 \\ 9x - 3x &= -11 - 7 \\ 6x &= -18 \\ x &= -3 \end{aligned}$$

7.
$$\begin{aligned} 3x - 8 &= 5x + 8 \\ 3x - 5x &= 8 + 8 \\ -2x &= 16 \\ x &= -8 \end{aligned}$$

8.
$$\begin{aligned} 5 - 6x &= 1 - 4x \\ 4x - 6x &= 1 - 5 \\ -2x &= -4 \\ x &= 2 \end{aligned}$$

9.
$$\begin{aligned} 5x + 14 &= 17 - 3x \\ 5x + 3x &= 17 - 14 \\ 8x &= 3 \\ x &= \frac{3}{8} \end{aligned}$$

10.
$$\begin{aligned} 3x - 31 &= 11x - 16 \\ 3x - 11x &= 31 - 16 \\ -8x &= 15 \\ x &= -\frac{15}{8} = -1\tfrac{7}{8} \end{aligned}$$

11.
$$\begin{aligned} 18 - 7x &= 18x - 7 \\ -7x - 18x &= -18 - 7 \\ -25x &= -25 \\ x &= 1 \end{aligned}$$

12.
$$\begin{aligned} 27 + 10x &= 13x + 23 \\ 10x - 13x &= 23 - 27 \\ -3x &= -4 \\ x &= \frac{4}{3} \end{aligned}$$

13.
$$\begin{aligned} 19x - 11 &= 15 + 6x \\ 19x - 6x &= 15 + 11 \\ 13x &= 26 \\ x &= 2 \end{aligned}$$

14.
$$\begin{aligned} 32x - 15 &= 7 + 65x \\ 32x - 65x &= 15 + 7 \\ -33x &= 22 \\ x &= -\frac{2}{3} \end{aligned}$$

15.
$$\begin{aligned} 13x - 81 &= 5x - 31x - 159 \\ 13x - 5x + 31x &= 81 - 159 \\ 39x &= -78 \\ x &= -2 \end{aligned}$$

16.

$$\begin{aligned}12x-20x+13&=9x-259\\12x-20x-9x&=-259-13\\-17x&=-272\\x&=16\end{aligned}$$

18.

$$\begin{aligned}3+2(2x+3)&=2x-3(2x+1)\\3+4x+6&=2x-6x-3\\4x-2x+6x&=-3-6-3.\\8x&=-12\\x&=-\frac{3}{2}\end{aligned}$$

19.

$$\begin{aligned}2x-(4x-1)&=5x-(x-1)\\2x-4x+1&=5x-x+1\\2x-4x-5x+x&=1-1\\-6x&=0\\x&=0\end{aligned}$$

20.

$$\begin{aligned}7(x-2)-5(x+3)&=3(2x-5)-6(4x-1)\\7x-14-5x-15&=6x-15-24x+6\\7x-5x-6x+24x&=14+15-15+6\\20x&=20\\x&=1\end{aligned}$$

21.

$$\begin{aligned}3(3x+5)-2(5x-3)&=13-(5x-16)\\9x+15-10x+6&=13-5x+16\\9x-10x+5x&=-15-6+13+16\\4x&=8\\x&=2\end{aligned}$$

22.

$$\begin{aligned}(2x-1)(3x+2)&=(3x-5)(2x+20)\\6x^2+x-2&=6x^2+50x-100\\x-50x&=2-100\\-49x&=-98\\x&=2\end{aligned}$$

23.

$$\begin{aligned}(5-6x)(2x-1)&=(3x+3)(13-4x)\\-12x^2+16x-5&=-12x^2+27x+39\\16x-27x&=5+39\\-11x&=44\\x&=-4\end{aligned}$$

24.

$$\begin{aligned}(x-3)^2-(5-x)^2&=-4x\\x^2-+6x\ 9-25+10x-x^2&=-4x\\-6x+10x+4x&=-9+25\\8x&=16\\x&=2\end{aligned}$$

25.

$$(2x-1)^2-3(x-2)+5(3x-2)-(5-2x)^2=0$$
$$4x^2-4x+1-3x+6+15x-10-25+20x-4x^2=0$$
$$-4x-3x+15x+20x=-1-6+10+25$$
$$28x=28$$
$$x=1$$

26.

$$2(x-2)^2-3(x-1)^2+x^2=1$$
$$2x^2-8x+8-3x^2+6x-3+x^2=1$$
$$-8x+6x=-8+3+1$$
$$-2x=-4$$
$$x=2$$

27.

$$(x-1)(x-2)(x+4)=(x+2)(x+3)(x-4)$$
$$x^3+x^2-10x+8=x^3+x^2-14x-24$$
$$-10x+14x=-8-24$$
$$4x=-32$$
$$x=-8$$

28.

$$5(7+3x)-(2x-3)(1-2x)-(2x-3)^2-(5+x)=0$$
$$35+15x+4x^2-8x+3-4x^2+12x-9-5-x=0$$
$$15x-8x+12x-x=-35-3+9+5$$
$$18x=-24$$
$$x=-\frac{4}{3}$$

29.

$$(5x-1)^2-(3x+2)^2-(4x-3)^2+4=0$$
$$25x^2-10x+1-9x^2-12x-4-16x^2+24x-9+4=0$$
$$-10x-12x+24x=-1+4+9-4$$
$$2x=8$$
$$x=4$$

30.

$$(2x+1)^3+(2x-1)^3=16x(x^2-4)-228$$
$$8x^3+12x^2+6x+1+8x^3-12x^2+6x-1=16x^3-64x-228$$
$$6x+6x+64x=-1+1-228$$
$$76x=-228$$
$$x=-3$$

Art. 175.—Pages 111–115.

2.

$$x+\frac{x}{2}+\frac{x}{3}=-11.$$

Multiplying each term by 6,

$$6x+3x+2x=-66$$
$$11x=-66$$
$$x=-6$$

3.

$$\frac{3x}{4}-\frac{5x}{6}+\frac{1}{18}=0.$$

Multiplying each term by 36,

$$27x-30x+2=0$$
$$-3x=-2$$
$$x=\frac{2}{3}$$

4.

$$2x-\frac{3x}{4}=\frac{13}{14}-\frac{x}{7}.$$

Multiplying each term by 28,

$$56x-21x=26-4x$$
$$56x-21x+4x=26$$
$$39x=26$$
$$x=\frac{2}{3}$$

5.

$$\frac{7x}{4}-7=\frac{5x}{3}-\frac{9x}{4}.$$

Multiplying each term by 12,

$$21x-84=20x-27x$$
$$21x-20x+27x=84$$
$$28x=84$$
$$x=3$$

6.

$$\frac{1}{6}+\frac{1}{2x}=\frac{1}{4}+\frac{1}{12x}.$$

Multiplying each term by $12x$,

$$2x+6=3x+1$$
$$2x-3x=1-6$$
$$-x=-5$$
$$x=5$$

7.

$$\frac{2x}{5}-\frac{9x}{20}-\frac{7x}{10}=\frac{5}{4}.$$

Multiplying each term by 20,

$$8x-9x-14x=25$$
$$-15x=25$$
$$x=-\frac{5}{3}=-1\tfrac{2}{3}$$

8.

$$\frac{3}{x}-\frac{5}{2x}=7-\frac{3}{2x}.$$

Multiplying each term by $2x$,

$$6-5=14x-3$$
$$-14x=-6+5-3$$
$$-14x=-4$$
$$x=\frac{2}{7}$$

9.

$$\frac{x}{2}+\frac{11}{6}-\frac{x}{3}=\frac{x}{6}-\frac{3x}{4}.$$

Multiplying each term by 12,

$$6x+22-4x=2x-9x$$
$$6x-4x-2x+9x=-22$$
$$9x=-22$$
$$x=-\frac{22}{9}=-2\tfrac{4}{9}$$

10.

$$x-\frac{x}{7}+20=\frac{x}{2}+\frac{x}{4}+26.$$

Multiplying each term by 28,

$$28x-4x+560=14x+7x+728$$
$$28x-4x-14x-7x=728-560$$
$$3x=168$$
$$x=56$$

11.

$$\frac{3}{x}-\frac{7}{2x}=\frac{7}{12}-\frac{5}{3x}.$$

Multiplying each term by $12x$,

$$36-42=7x-20$$
$$-7x=-36+42-20$$
$$-7x=-14$$
$$x=2$$

13.

$$3x+\frac{5x+3}{7}=\frac{7x}{2}.$$

Multiplying each term by 14,

$$42x+10x+6=49x$$
$$42x+10x-49x=-6$$
$$3x=-6$$
$$x=-2$$

14.

$$x-\frac{2x+1}{5}=5x-\frac{5}{3}.$$

Multiplying each term by 15,

$$15x-(6x+3)=75x-25$$
$$15x-6x-3=75x-25$$
$$15x-6x-75x=3-25$$
$$-66x=-22$$
$$x=\frac{1}{3}$$

15.

$$7x-\frac{11x-3}{4}=3x+7.$$

Multiplying each term by 4,

$$28x-(11x-3)=12x+28$$
$$28x-11x+3=12x+28$$
$$28x-11x-12x=28-3$$
$$5x=25$$
$$x=5$$

16.

$$4x-\frac{2x-3}{3}+\frac{1}{2}(x-9)=5x.$$

Multiplying each term by 6,

$$24x-(4x-6)+3(x-9)=30x$$
$$24x-4x+6+3x-27=30x$$
$$24x-4x+3x-30x=27-6$$
$$-7x=21$$
$$x=-3$$

17.

$$x-(3x-4)-\frac{5-2x}{4}=2.$$

Multiplying each term by 4,

$$4x-(12x-16)-(5-2x)=8$$
$$4x-12x+16-5+2x=8$$
$$4x-12x+2x=-16+5+8$$
$$-6x=-3$$
$$x=\frac{1}{2}$$

18.

$$\frac{2x}{21}=x-7+\frac{x+3}{15}.$$

Multiplying each term by 105,

$$10x=105x-735+7x+21$$
$$10x-105x-7x=-735+21$$
$$-102x=-714$$
$$x=7$$

19.

$$\frac{x+1}{2}-\frac{x+4}{5}=\frac{x-4}{7}.$$

Multiplying each term by 70,

$$35x+35-(14x+56)=10x-40$$
$$35x+35-14x-56=10x-40$$
$$35x-14x-10x=-35+56-40$$
$$11x=-19$$
$$x=-\frac{19}{11}=-1\tfrac{8}{11}$$

20.

$$2-\frac{7x-1}{6}=3x-\frac{19x+3}{4}.$$

Multiplying each term by 12,

$$24-(14x-2)=36x-(57x+9)$$
$$24-14x+2=36x-57x-9$$
$$-14x-36x+57x=-24-2-9$$
$$7x=-35$$
$$x=-5$$

21.

$$\frac{5x-2}{3}-\frac{3x+4}{4}-\frac{7x+2}{6}=\frac{x-10}{2}.$$

Multiplying each term by 12,

$$20x-8-(9x+12)-(14x+4)=6x-60$$
$$20x-8-9x-12-14x-4=6x-60$$
$$20x-9x-14x-6x=8+12+4-60$$
$$-9x=-36$$
$$x=4$$

22.

$$\frac{1}{2}(x+1)-\frac{2x-5}{5}=\frac{11x+5}{10}-\frac{x-13}{3}.$$

Multiplying each term by 30,

$$15(x+1)-(12x-30)=33x+15-(10x-130)$$
$$15x+15-12x+30=33x+15-10x+130$$
$$15x-12x-33x+10x=-30+130$$
$$-20x=100$$
$$x=-5$$

23.

$$\frac{5x+1}{3}+\frac{17x+7}{9}-\frac{1}{2}(3x-1)=\frac{7x-1}{6}.$$

Multiplying each term by 18,

$$30x+6+34x+14-9(3x-1)=21x-3$$
$$30x+6+34x+14-27x+9=21x-3$$
$$30x+34x-27x-21x=-6-14-9-3$$
$$16x=-32$$
$$x=-2$$

24.

$$\frac{4+x}{7}=\frac{1}{2}(3x-2)-\frac{11x+2}{14}-\frac{1}{3}(2-9x).$$

Multiplying each term by 42,

$$24+6x=21(3x-2)-(33x+6)-14(2-9x)$$
$$24+6x=63x-42-33x-6-28+126x$$
$$6x-63x+33x-126x=-24-42-6-28$$
$$-150x=-100$$
$$x=\frac{2}{3}$$

25.

$$\frac{2x+1}{3}=\frac{4x+5}{4}-\frac{8+x}{6}+\frac{2x+5}{8}.$$

Multiplying each term by 24,

$$16x+8=24x+30-(32+4x)+6x+15$$
$$16x+8=24x+30-32-4x+6x+15$$
$$16x-24x+4x-6x=-8+30-32+15$$
$$-10x=5$$
$$x=-\frac{1}{2}$$

26.

$$\frac{5x-1}{2}-\frac{7-3x}{3x}=\frac{10x-3}{4}-\frac{3-5x}{2x}.$$

Multiplying each term by $12x$,

$$30x^2-6x-(28-12x)=30x^2-9x-(18-30x)$$
$$30x^2-6x-28+12x=30x^2-9x-18+30x$$
$$-6x+12x+9x-30x=28-18$$
$$-15x=10$$
$$x=-\frac{2}{3}$$

27.

$$\frac{3x+7}{2}-\frac{4(x^2-2)}{3x}-\frac{x^3+16}{6x^2}=\frac{7}{2}.$$

Multiplying each term by $6x^2$,

$$3x^2(3x+7)-8x(x^2-2)-(x^3+16)=21x^2$$
$$9x^3+21x^2-8x^3+16x-x^3-16=21x^2$$
$$\therefore\ 16x=16$$
$$x=1$$

30.

$$\frac{1}{3x-7}-\frac{2}{3x+7}=0.$$

Clearing of fractions,

$$3x+7-2(3x-7)=0$$
$$3x+7-6x+14=0$$
$$3x-6x=-7-14$$
$$-3x=-21$$
$$x=7$$

31.

$$\frac{2x-1}{3x+4}=\frac{2x+7}{3x+2}.$$

Clearing of fractions,

$$(2x-1)(3x+2)=(3x+4)(2x+7)$$
$$6x^2+x-2=6x^2+29x+28$$
$$x-29x=2+28$$
$$-28x=30$$
$$x=-\frac{15}{14}=-1\tfrac{1}{14}$$

32.

$$\frac{6x^2-7x+5}{2x^2+5x-13}=3.$$

Clearing of fractions,

$$6x^2-7x+5=6x^2+15x-39$$
$$-7x-15x=-5-39$$
$$-22x=-44$$
$$x=2$$

33.

$$\frac{5x-2}{x(x-1)}=\frac{5x+7}{x^2-1}.$$

Multiplying each term by $x(x+1)(x-1)$,

$$(5x-2)(x+1)=x(5x+7)$$
$$5x^2+3x-2=5x^2+7x$$
$$3x-7x=2$$
$$-4x=2$$
$$x=-\frac{1}{2}$$

34.

$$\frac{x}{3}-\frac{x^2-5x}{3x-7}=\frac{2}{3}.$$

Multiplying each term by 3,

$$x-\frac{3x^2-15x}{3x-7}=2$$
$$x-2=\frac{3x^2-15x}{3x-7}.$$

Multiplying each term by $3x-7$,

$$(x-2)(3x-7)=3x^2-15x$$
$$3x^2-13x+14=3x^2-15x$$
$$-13x+15x=-14$$
$$2x=-14$$
$$x=-7$$

35.

$$\frac{(x+5)^2}{x-3} = \frac{5x+1}{5}.$$

Clearing of fractions,

$$5(x+5)^2 = (x-3)(5x+1)$$
$$5x^2 + 50x + 125 = 5x^2 - 14x - 3$$
$$50x + 14x = -125 - 3$$
$$64x = -128$$
$$x = -2$$

36.

$$\frac{1}{x+1} + \frac{2}{x+2} = \frac{3}{x+3}.$$

Clearing of fractions,

$$(x+2)(x+3) + 2(x+1)(x+3) = 3(x+1)(x+2)$$
$$x^2 + 5x + 6 + 2x^2 + 8x + 6 = 3x^2 + 9x + 6$$
$$x^2 + 2x^2 - 3x^2 + 5x + 8x - 9x = -6$$
$$4x = -6$$
$$x = -\frac{3}{2}$$

37.

$$\frac{3x+2}{6} - \frac{2x-1}{3x-7} = \frac{x}{2}.$$

Multiplying each term by 6,

$$3x + 2 - \frac{12x-6}{3x-7} = 3x$$
$$2 = \frac{12x-6}{3x-7}$$

Dividing by 2,

$$1 = \frac{6x-3}{3x-7}$$

Clearing of fractions,

$$3x - 7 = 6x - 3$$
$$3x - 6x = 7 - 3$$
$$-3x = 4$$
$$x = -\frac{4}{3}$$

38.

$$\frac{2}{x-2} - \frac{1}{x-3} = \frac{1}{x^2-5x+6}.$$

Multiplying each term by $(x-2)(x-3)$, or x^2-5x+6,

$$2(x-3) - (x-2) = 1$$
$$2x - 6 - x + 2 = 1$$
$$2x - x = 6 - 2 + 1$$
$$x = 5$$

39.

$$\frac{6x+7}{15}-\frac{2(x-1)}{7x-6}=\frac{2x+1}{5}.$$

Multiplying each term by 15,

$$6x+7-\frac{30(x-1)}{7x-6}=6x+3$$

$$7-3=\frac{30(x-1)}{7x-6}$$

$$4=\frac{30(x-1)}{7x-6}$$

Dividing by 2, $\quad 2=\frac{15(x-1)}{7x-6}$

Clearing of fractions,

$$14x-12=15x-15$$

$$14x-15x=12-15$$

$$-x=-3$$

$$x=3$$

40.

$$\frac{3}{1-x}-\frac{2}{1+x}-\frac{1}{1-x^2}=0.$$

Multiplying each term by $(1+x)(1-x)$,

$$3(1+x)-2(1-x)-1=0$$

$$3+3x-2+2x-1=0$$

$$3x+2x=0$$

$$5x=0$$

$$x=0$$

41.

$$\frac{2x^2+3x}{2x+1}+\frac{1}{3x}=x+1.$$

Clearing of fractions,

$$3x(2x^2+3x)+2x+1=3x(x+1)(2x+1)$$

$$6x^3+9x^2+2x+1=6x^3+9x^2+3x$$

$$2x-3x=-1$$

$$-x=-1$$

$$x=1$$

42.

$$2\left(\frac{x+1}{x+2}\right)+3\left(\frac{x+2}{x+1}\right)=5.$$

Clearing of fractions,

$$2(x+1)^2+3(x+2)^2=5(x+1)(x+2)$$

$$2x^2+4x+2+3x^2+12x+12=5x^2+15x+10$$

$$2x^2+3x^2-5x^2+4x+12x-15x=-2-12+10$$

$$x=-4$$

43.

$$\frac{6}{3x+1}-\frac{1}{x+1}=\frac{2}{2x-1}.$$

Clearing of fractions,

$$\begin{aligned}
6(x+1)(2x-1)-(3x+1)(2x-1) &= 2(3x+1)(x+1)\\
12x^2+6x-6-(6x^2-x-1) &= 6x^2+8x+2\\
12x^2+6x-6-6x^2+x+1 &= 6x^2+8x+2\\
12x^2-6x^2-6x^2+6x+x-8x &= 6-1+2\\
-x &= 7\\
x &= -7
\end{aligned}$$

44.

$$\frac{x}{9}=\frac{x+1}{3}-\frac{7-2x^2}{1-9x}.$$

Multiplying each term by 9,

$$\begin{aligned}
x &= 3x+3-\frac{63-18x^2}{1-9x}\\
\frac{63-18x^2}{1-9x} &= 3x+3-x=2x+3
\end{aligned}$$

Clearing of fractions,

$$\begin{aligned}
63-18x^2 &= (2x+3)(1-9x)\\
63-18x^2 &= -18x^2-25x+3\\
25x &= -63+3\\
25x &= -60\\
x &= -\frac{12}{5}=-2\tfrac{2}{5}
\end{aligned}$$

45.

$$\frac{(x+1)^2}{(x+2)^2}=\frac{x-4}{x-2}.$$

Clearing of fractions,

$$\begin{aligned}
(x-2)(x+1)^2 &= (x-4)(x+2)^2\\
(x-2)(x^2+2x+1) &= (x-4)(x^2+4x+4)\\
x^3-3x-2 &= x^3-12x-16\\
-3x+12x &= 2-16\\
9x &= -14\\
x &= -\frac{14}{9}=-1\tfrac{5}{9}
\end{aligned}$$

46.

$$\frac{2x^2-3x+2}{3x^2+x-1}=\frac{2x-3}{3x+1}.$$

Clearing of fractions,

$$(3x+1)(2x^2-3x+2)=(2x-3)(3x^2+x-1)$$
$$6x^3-7x^2+3x+2=6x^3-7x^2-5x+3$$
$$3x+5x=-2+3$$
$$8x=1$$
$$x=\frac{1}{8}$$

47.

$$\frac{x-1}{x-2}+\frac{x+1}{x+2}=\frac{2(x^2+4x+1)}{(x+2)^2}.$$

Multiplying each term by $(x-2)(x+2)^2$,

$$(x-1)(x+2)^2+(x+1)(x-2)(x+2)=2(x-2)(x^2+4x+1)$$
$$(x-1)(x^2+4x+4)+(x+1)(x^2-4)=2(x-2)(x^2+4x+1)$$
$$x^3+3x^2-4+x^3+x^2-4x-4=2x^3+4x^2-14x-4$$
$$x^3+x^3-2x^3+3x^2+x^2-4x^2-4x+14x=4$$
$$10x=4$$
$$x=\frac{2}{5}$$

48.

$$\frac{4x+3}{10}-\frac{12x-5}{5x-29}-\frac{2x-1}{5}=0.$$

Multiplying each term by 10,

$$4x+3-\frac{120x-50}{5x-29}-(4x-2)=0$$
$$4x+3-\frac{120x-50}{5x-29}-4x+2=0$$
$$5=\frac{120x-50}{5x-29}$$

Dividing by 5, $\quad 1=\dfrac{24x-10}{5x-29}$

Clearing of fractions,

$$5x-29=24x-10$$
$$5x-24x=29-10$$
$$-19x=19$$
$$x=-1$$

49.

$$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-3}{x-4}-\frac{x-4}{x-5}.$$

Adding the fractions in the first member, and the fraction in the second member,

$$\frac{(x-1)(x-3)-(x-2)^2}{(x-2)(x-3)}=\frac{(x-3)(x-5)-(x-4)^2}{(x-4)(x-5)}$$

$$\frac{x^2-4x+3-x^2+4x-4}{(x-2)(x-3)}=\frac{x^2-8x+15-x^2+8x-16}{(x-4)(x-5)}$$

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-4)(x-5)}$$

Clearing of fractions, and changing signs,

$$(x-4)(x-5)=(x-2)(x-3)$$

$$x^2-9x+20=x^2-5x+6$$

$$-9x+5x=-20+6$$

$$-4x=-14$$

$$x=\frac{7}{2}$$

Art. 176.—Pages 116, 117.

3.

$$2ax+d=3c-bx.$$

$$2ax+bx=3c-d$$

$$x(2a+b)=3c-d$$

$$x=\frac{3c-d}{2a+b}$$

4.

$$6bmx-5an=15am-2bnx.$$

$$6bmx+2bnx=15am+5an$$

$$2bx(3m+n)=5a(3m+n)$$

$$x=\frac{5a(3m+n)}{2b(3m+n)}=\frac{5a}{2b}$$

5.

$$x+1=2ax-a^2(x-1).$$

$$x+1=2ax-a^2x+a^2$$

$$a^2x-2ax+x=a^2-1$$

$$x(a-1)^2=(a+1)(a-1)$$

$$x=\frac{(a+1)(a-1)}{(a-1)^2}=\frac{a+1}{a-1}$$

6.

$$\frac{a^2}{x}+\frac{b}{2}=\frac{4b^2}{x}+\frac{a}{4}.$$

Multiplying each term by $4x$,

$$4a^2+2bx=16b^2+ax$$

$$2bx-ax=16b^2-4a^2$$

$$x(2b-a)=4(2b+a)(2b-a)$$

$$x=4(2b+a)=8b+4a$$

7.

$$(a^2-2x)^2=(4x-3a^2)(x+a^2).$$

$$a^4-4a^2x+4x^2=4x^2+a^2x-3a^4$$

$$-4a^2x-a^2x=-a^4-3a^4$$

$$-5a^2x=-4a^4$$

$$x=\frac{4a^4}{5a^2}=\frac{4a^2}{5}$$

8.

$$(2m + 3x)(2m - 3x) = n^2 - (3x - n)^2.$$
$$4m^2 - 9x^2 = n^2 - 9x^2 + 6nx - n^2$$
$$-6nx = -4m^2$$
$$x = \frac{4m^2}{6n} = \frac{2m^2}{3n}$$

9.

$$\frac{x-a}{b} - \frac{x+b}{a} + 2 = 0.$$

Clearing of fractions,

$$a(x-a) - b(x+b) + 2ab = 0$$
$$ax - a^2 - bx - b^2 + 2ab = 0$$
$$ax - bx = a^2 - 2ab + b^2$$
$$x(a-b) = (a-b)^2$$
$$x = a - b$$

10.

$$(x - a - b)^2 - (x-a)(x-b) + ab = 0.$$
$$x^2 - 2ax - 2bx + a^2 + 2ab + b^2 - x^2 + ax + bx - ab + ab = 0$$
$$-2ax + ax - 2bx + bx = -a^2 - 2ab - b^2$$

Uniting terms and changing signs,

$$ax + bx = a^2 + 2ab + b^2$$
$$x(a+b) = (a+b)^2$$
$$x = a + b$$

11.

$$\frac{x}{x-a} - \frac{x+2b}{x+a} = \frac{a^2+b^2}{x^2-a^2}.$$

Multiplying each term by $x^2 - a^2$,

$$x(x+a) - (x-a)(x+2b) = a^2 + b^2$$
$$x^2 + ax - x^2 + ax - 2bx + 2ab = a^2 + b^2$$
$$2ax - 2bx = a^2 - 2ab + b^2$$
$$2x(a-b) = (a-b)^2$$
$$x = \frac{(a-b)^2}{2(a-b)} = \frac{a-b}{2}$$

12.
$$\frac{(b-3x)(c+2x)}{2(x-c)(b-3c-3x)}=1.$$

Clearing of fractions,

$$(b-3x)(c+2x)=2(x-c)(b-3c-3x)$$
$$bc+2bx-3cx-6x^2=-2bc+2bx+6c^2-6x^2$$
$$-3cx=-bc-2bc+6c^2$$

Uniting terms and changing signs,

$$3cx=3bc-6c^2$$
$$3cx=3c(b-2c)$$
$$x=b-2c$$

13.
$$(x+a)^3-(x-a)^3-a(3x-a)(2x+a)=x(a+1)+3.$$
$$x^3+3ax^2+3a^2x+a^3-x^3+3ax^2-3a^2x+a^3-6ax^2-a^2x+a^3=ax+x+3$$
$$-a^2x-ax-x=-a^3-a^3-a^3+3$$

Uniting terms and changing signs,

$$a^2x+ax+x=3a^3-3$$
$$x(a^2+a+1)=3(a-1)(a^2+a+1)$$
$$x=3(a-1)=3a-3$$

14.
$$\frac{(n^2-x^2)(n+x)}{x+2n}=-x^2+nx+n^2.$$

Clearing of fractions,

$$(n^2-x^2)(n+x)=(x+2n)(-x^2+nx+n^2)$$
$$n^3+n^2x-nx^2-x^3=-x^3-nx^2+3n^2x+2n^3$$
$$n^2x-3n^2x=-n^3+2n^3$$
$$-2n^2x=n^3$$
$$x=-\frac{n^3}{2n^2}=-\frac{n}{2}$$

15.
$$(a-x)(b-x)-a(b+1)=\frac{a^2}{b}+x^2.$$

Clearing of fractions,

$$b(a-x)(b-x)-ab(b+1)=a^2+bx^2$$
$$ab^2-abx-b^2x+bx^2-ab^2-ab=a^2+bx^2$$
$$-abx-b^2x=a^2+ab$$
$$-bx(a+b)=a(a+b)$$
$$x=-\frac{a(a+b)}{b(a+b)}=-\frac{a}{b}$$

16.

$$\frac{x}{2a} - 3 + \frac{x}{4a^3} = \frac{x}{3a^2} - 2a(2-3a).$$

Multiplying each term by $12a^3$,

$$6a^2x - 36a^3 + 3x = 4ax - 24a^4(2-3a)$$
$$6a^2x - 36a^3 + 3x = 4ax - 48a^4 + 72a^5$$
$$6a^2x - 4ax + 3x = 72a^5 - 48a^4 + 36a^3$$
$$x(6a^2 - 4a + 3) = 12a^3(6a^2 - 4a + 3)$$
$$x = 12a^3$$

17.

$$\frac{x}{2} + \frac{1-2ax}{2a} + \frac{2x-1}{a^2} = 0.$$

Multiplying each term by $2a^2$,

$$a^2x + a - 2a^2x + 4x - 2 = 0$$
$$a^2x - 2a^2x + 4x = -a + 2$$

Uniting terms and changing signs,

$$a^2x - 4x = a - 2$$
$$x(a+2)(a-2) = a - 2$$
$$x = \frac{a-2}{(a+2)(a-2)} = \frac{1}{a+2}$$

18.

$$\frac{x}{mn} - \frac{x+mn}{3n} = \frac{x}{3n} - (m-1).$$

Multiplying each term by $3mn$,

$$3x - m(x+mn) = mx - 3mn(m-1)$$
$$3x - mx - m^2n = mx - 3m^2n + 3mn$$
$$3x - mx - mx = 3mn - 3m^2n + m^2n$$
$$3x - 2mx = 3mn - 2m^2n$$
$$x(3-2m) = mn(3-2m)$$
$$x = mn$$

19.

$$\frac{x+2a}{x-a} + \frac{x-3a}{x+a} = 2.$$

Clearing of fractions,

$$(x+a)(x+2a) + (x-a)(x-3a) = 2(x+a)(x-a)$$
$$x^2 + 3ax + 2a^2 + x^2 - 4ax + 3a^2 = 2x^2 - 2a^2$$
$$x^2 + x^2 - 2x^2 + 3ax - 4ax = -2a^2 - 3a^2 - 2a^2$$
$$-ax = -7a^2$$
$$x = \frac{7a^2}{a} = 7a$$

20.

$$\frac{4x-a}{2x-a}-\frac{x+a}{x-a}=1.$$

Clearing of fractions,

$$(x-a)(4x-a)-(x+a)(2x-a)=(x-a)(2x-a)$$
$$4x^2-5ax+a^2-2x^2-ax+a^2=2x^2-3ax+a^2$$
$$4x^2-2x^2-2x^2-5ax-ax+3ax=-a^2$$
$$-3ax=-a^2$$
$$x=\frac{a^2}{3a}=\frac{a}{3}$$

21.

$$\frac{x}{2}-\frac{a-bcx}{2bc}=\frac{x}{6c}-\frac{ac-4bx}{3bc}.$$

Multiplying each term by $6bc$,

$$3bcx-(3a-3bcx)=bx-2(ac-4bx)$$
$$3bcx-3a+3bcx=bx-2ac+8bx$$
$$3bcx+3bcx-bx-8bx=-2ac+3a$$
$$6bcx-9bx=-2ac+3a$$
$$3bx(2c-3)=-a(2c-3)$$
$$x=-\frac{a(2c-3)}{3b(2c-3)}=-\frac{a}{3b}$$

22.

$$\frac{ax+b}{ax-b}-\frac{3b}{ax+b}=\frac{a^2x^2+b^2}{a^2x^2-b^2}.$$

Multiplying each term by $(ax+b)(ax-b)$,

$$(ax+b)^2-3b(ax-b)=a^2x^2+b^2$$
$$a^2x^2+2abx+b^2-3abx+3b^2=a^2x^2+b^2$$
$$2abx-3abx=-3b^2$$
$$-abx=-3b^2$$
$$x=\frac{3b^2}{ab}=\frac{3b}{a}$$

23.

$$\frac{ax-b}{ax+b}-\frac{bx-a}{bx+a}=\frac{a-b}{(ax+b)(bx+a)}.$$

Multiplying each term by $(ax+b)(bx+a)$,

$$(ax-b)(bx+a)-(ax+b)(bx-a)=a-b$$
$$abx^2+a^2x-b^2x-ab-abx^2+a^2x-b^2x+ab=a-b$$
$$a^2x+a^2x-b^2x-b^2x=a-b$$
$$2a^2x-2b^2x=a-b$$
$$2x(a+b)(a-b)=a-b$$
$$x=\frac{a-b}{2(a+b)(a-b)}$$
$$x=\frac{1}{2(a+b)}$$

24.

$$\frac{x-n}{m}-\frac{x^2-mx-n^2}{mx-n^2}=1+\frac{n^2}{mx-n^2}.$$

Multiplying each term by $m(mx-n^2)$,

$$(x-n)(mx-n^2)-m(x^2-mx-n^2)=m(mx-n^2)+mn^2$$
$$mx^2-mnx-n^2x+n^3-mx^2+m^2x+mn^2=m^2x-mn^2+mn^2$$
$$-mnx-n^2x=-mn^2-n^3$$

Changing signs,

$$mnx+n^2x=mn^2+n^3$$
$$nx(m+n)=n^2(m+n)$$
$$x=\frac{n^2(m+n)}{n(m+n)}=n$$

Art. 177.—Page 118.

2.

$$.23x-2.05=.02x-1.882.$$
$$.23x-.02x=2.05-1.882$$
$$.21x=.168$$
$$x=\frac{.168}{.21}=.8$$

3.

$$.001x-.32=.09x-.2x-.653.$$
$$.001x-.09x+.2x=.32-.653$$
$$.111x=-.333$$
$$x=-\frac{.333}{.111}=-3$$

4.

$$.3x-.02-.003x=.7-.06x-.006.$$
$$.3x-.003x+.06x=.7-.006+.02$$
$$.357x=.714$$
$$x=\frac{.714}{.357}=2$$

5.

$$.3(1.2x-5)=14+.05x.$$
$$.36x-1.5=14+.05x$$
$$.36x-.05x=14+1.5$$
$$.31x=15.5$$
$$x=\frac{15.5}{.31}=50$$

6.

$$.7(x+.13)=.03(4x-.1)+.5.$$
$$.7x+.091=.12x-.003+.5$$
$$.7x-.12x=-.091-.003+.5$$
$$.58x=.406$$
$$x=\frac{.406}{.58}=.7$$

7.

$$3.3x-\frac{.72x-.55}{.5}=.1x+9.9.$$

Clearing of fractions,

$$1.65x-.72x+.55=.05x+4.95$$
$$1.65x-.72x-.05x=4.95-.55$$
$$.88x=4.4$$
$$x=\frac{4.4}{.88}=5$$

8.
$$4.25 - \frac{.2}{x} = \frac{17}{4} - \frac{1 - .1x}{x}.$$

Multiplying each term by $4x$,

$$17x - .8 = 17x - (4 - .4x)$$
$$17x - .8 = 17x - 4 + .4x$$
$$-.4x = .8 - 4$$
$$-.4x = -3.2$$
$$x = \frac{3.2}{.4} = 8$$

9.
$$\frac{.6x + .044}{.4} - \frac{.5x - .178}{.6} = .38.$$

Multiplying each term by 1.2,

$$3(.6x + .044) - 2(.5x - .178) = .456$$
$$1.8x + .132 - x + .356 = .456$$
$$1.8x - x = .456 - .132 - .356$$
$$.8x = -.032$$
$$x = -\frac{.032}{.8} = -.04$$

10.
$$\frac{2 - 3x}{1.5} + \frac{5x}{1.25} - \frac{2x - 3}{9} = \frac{x - 2}{1.8} + \frac{25}{9}.$$

Multiplying each term by 9,

$$6(2 - 3x) + 36x - (2x - 3) = 5(x - 2) + 25$$
$$12 - 18x + 36x - 2x + 3 = 5x - 10 + 25$$
$$-18x + 36x - 2x - 5x = 25 - 10 - 12 - 3$$
$$11x = 0$$
$$x = 0$$

CHAPTER XIII.

Art. 179.—Pages 120–131.

4. Let $x =$ the number.

Then, $2x =$ its double,

and $\frac{x}{2} =$ its half.

By the conditions, $2x - \frac{x}{2} = 45$

$$4x - x = 90$$
$$3x = 90.$$

Whence, $x = 30$, the number required.

5. Let $x =$ one part.

Then, $34 - x =$ the other part.

By the conditions, $\frac{4x}{7} = \frac{2}{5}(34 - x)$

$$20x = 14(34 - x)$$
$$20x = 476 - 14x$$
$$34x = 476.$$

Whence, $x = 14$, one part,

and $34 - x = 20$, the other part

6. Let $x =$ the number.

Then, $\frac{x}{3} =$ its third part,

$\frac{x}{10} =$ its tenth part,

and $\frac{x}{12} =$ its twelfth part.

By the conditions,

$$x - \left(\frac{x}{3} + \frac{x}{10} + \frac{x}{12}\right) = 58$$
$$x - \frac{x}{3} - \frac{x}{10} - \frac{x}{12} = 58$$
$$60x - 20x - 6x - 5x = 3480$$
$$29x = 3480.$$

Whence, $x = 120$, the number required.

7. Let $x =$ the greater part.

Then, $59 - x =$ the less.

By the conditions, $\frac{x}{7} + \frac{59 - x}{3} = 13$

$$3x + 413 - 7x = 273$$

$$-4x = -140.$$

Whence, $x = 35$, the greater part,

and $59 - x = 24$, the less.

8. Let $x =$ B's age.

Then, $4x =$ A's age.

Also, $x + 30 =$ B's age 30 years hence,

and $4x + 30 =$ A's age 30 years hence.

By the conditions, $4x + 30 = 2(x + 30)$

$$4x + 30 = 2x + 60$$

$$2x = 30.$$

Whence, $x = 15$, B's age,

and $4x = 60$, A's age.

9. Let $x =$ the number of years.

Then, $62 - x =$ A's age x years ago,

and $36 - x =$ B's age x years ago.

By the conditions, $62 - x = 3(36 - x)$

$$62 - x = 108 - 3x$$

$$2x = 46.$$

Whence, $x = 23$, the number of years required.

10. Let $x =$ what A had at first.

Then, $2x =$ what B had at first.

After giving A \$42, B had left $2x - 42$ dollars, while A had $x + 42$ dollars. Then, by the conditions,

$$x + 42 = 4(2x - 42)$$

$$x + 42 = 8x - 168$$

$$-7x = -210.$$

Whence, $x = 30$, what A had at first,

and $2x = 60$, what B had at first.

11. Let $x =$ the greater part.

Then, $207 - x =$ the less.

By the conditions, $\frac{x}{4} = \frac{2}{7}(207 - x) + 3$

$$7x = 1656 - 8x + 84$$

$$15x = 1740.$$

Whence, $x = 116$, the greater part,

and $207 - x = 91$, the less.

12. Let $x =$ the less number.

Then, $x + 3 =$ the greater.

By the conditions, $(x+3)^2 - x^2 = 51$

$$x^2 + 6x + 9 - x^2 = 51$$
$$6x = 42.$$

Whence, $x = 7$, the less number,

and $x + 3 = 10$, the greater.

13. Let $x =$ the number of oxen.

Then, $2x =$ the number of cows.

By the conditions,

$$55x + 32(2x) = 1428$$
$$55x + 64x = 1428$$
$$119x = 1428$$

Whence, $x = 12$, the number of oxen,

and $2x = 24$, the number of cows.

14. Let $x =$ the greater part.

Then, $80 - x =$ the less.

By the conditions, $62 - x = 48 - (80 - x)$

$$62 - x = 48 - 80 + x$$
$$-2x = -94.$$

Whence, $x = 47$, the greater part,

and $80 - x = 33$, the less.

15. Let $x =$ what each son received.

Then, $2x =$ what each daughter received,

and $6x =$ what the wife received.

By the conditions,

$$6x + 3x + 2(2x) = 1872$$
$$6x + 3x + 4x = 1872$$
$$13x = 1872.$$

Whence, $x = 144$, what each son received.

Then, $2x = 288$, what each daughter received,

and $6x = 864$, what the wife received.

16. Let $x =$ B's share.

Then, $\frac{3x}{8} =$ A's share,

and $\frac{2}{9}$ of $\frac{3x}{8} = \frac{x}{12} =$ C's share.

By the conditions, $x + \frac{3x}{8} + \frac{x}{12} = 70$

$$24x + 9x + 2x = 1680$$
$$35x = 1680.$$

Whence, $x = 48$, B's share.

Then, $\frac{8x}{8} = 18$, A's share,

and $\frac{x}{12} = 4$, C's share.

17. Let $x =$ the number of artillery.
Then, $2x =$ the number of cavalry,
and $12\frac{1}{2} \times 2x = 25x =$ the number of infantry.
By the conditions, $x + 2x + 25x = 2744$

$$28x = 2744.$$

Whence, $x = 98$, the number of artillery.
Then, $2x = 196$, the number of cavalry,
and $25x = 2450$, the number of infantry.

18. Let $x =$ B's age.
Then, $x + 34 =$ A's age.
By the conditions, $x + 34 - 50 = 40 - x$

$$2x = 56.$$

Whence, $x = 28$, B's age,
and $x + 34 = 62$, A's age.

19. Let $x =$ the number of miles by water.

Then, $\frac{4x}{7} =$ the number on foot,

and $\frac{2x}{5} =$ the number on horseback.

By the conditions,

$$x + \frac{4x}{7} + \frac{2x}{5} = 3036$$

$$35x + 20x + 14x = 106260$$

$$69x = 106260.$$

Whence, $x = 1540$, the number of miles by water.

Then, $\frac{4x}{7} = 880$, the number on foot,

and $\frac{2x}{5} = 616$, the number on horseback.

21. Let $x =$ the first part.
Then, $a - x =$ the second.

By the conditions, $mx = \frac{a-x}{n}$

$$mnx = a - x \qquad (1)$$

$$x + mnx = a$$

$$x(1 + mn) = a$$

Whence, $x = \frac{a}{1 + mn}$, the first part.

Therefore by (1),

$$a - x = mnx = \frac{amn}{1 + mn}, \text{ the second part.}$$

22. Let $x =$ the least number.

Then $x + 1$, $x + 2$, and $x + 3$ will represent the others; hence, by the conditions,

$$x + x + 1 + x + 2 + x + 3 = 94$$
$$4x = 88.$$

Whence, $x = 22$, the least number.

Then, $x + 1 = 23$,

$x + 2 = 24$,

and $x + 3 = 25$.

Therefore the numbers are 22, 23, 24, and 25.

23. Let $x =$ the less part.

Then, $43 - x =$ the greater.

By the conditions, $43 - x - 20 = 3(17 - x)$

$$23 - x = 51 - 3x$$
$$2x = 28.$$

Whence, $x = 14$, the less part,

and $43 - x = 29$, the greater.

24. Let $x =$ A's share.

Then, $27 - x =$ B's share,

$25 - x =$ C's share,

and $23 - x =$ D's share

By the conditions,

$$x + 27 - x + 25 - x + 23 - x = 47$$
$$-2x = -28.$$

Whence, $x = 14$, A's share.

Then, $27 - x = 13$, B's share,

$25 - x = 11$, C's share,

and $23 - x = 9$, D's share.

25. Let $x =$ the number.

By the conditions, $80 - \frac{x + 15}{2} = x - 100$

$$160 - x - 15 = 2x - 200$$
$$-3x = -345.$$

Whence, $x = 115$, the number required.

26. Let $x =$ the fourth part.

Then, $4x =$ the third part,

$3 \times 4x = 12x =$ the second part,

and $2 \times 12x = 24x =$ the first part.

By the conditions,

$$x + 4x + 12x + 24x = 205$$
$$41x = 205.$$

Whence, $x = 5$, the fourth part.

Then, $4x = 20$, the third part,

$12x = 60$, the second part,

and $24x = 120$, the first part.

27. Let $x =$ B's age eleven years ago.

Then, $4x =$ A's age eleven years ago.

Also, $x + 24 =$ B's age thirteen years hence,

and $4x + 24 =$ A's age thirteen years hence.

By the conditions, $4x + 24 = 2(x + 24)$

$$4x + 24 = 2x + 48$$
$$2x = 24.$$

Whence, $x = 12$, B's age eleven years ago,

and $4x = 48$, A's age eleven years ago.

Therefore, $x + 11 = 23$, B's age at present,

and $4x + 11 = 59$, A's age at present.

28. Let $x =$ the less number.

Then, $x + 1 =$ the greater.

By the conditions,

$$(x + 1)^2 - x^2 + 3(x + 1) = x + 92$$
$$x^2 + 2x + 1 - x^2 + 3x + 3 = x + 92$$
$$4x = 88.$$

Whence, $x = 22$, the less number,

and $x + 1 = 23$, the greater number.

29. Let $x =$ the number.

By the conditions, $\frac{5x}{6} - 25 = 9 - \frac{x}{9}$

$$15x - 450 = 162 - 2x$$
$$17x = 612.$$

Whence, $x = 36$, the number required.

30. Let $x =$ B's age.

Then, $mx =$ A's age.

Also, $x + a =$ B's age in a years,

and $mx + a =$ A's age in a years.

By the conditions, $mx + a = n(x + a)$

$$mx + a = nx + na$$
$$mx - nx = na - a$$
$$x(m - n) = a(n - 1).$$

Whence, $x = \frac{a(n-1)}{m-n}$, B's age,

and $mx = \frac{am(n-1)}{m-n}$, A's age.

31. Let $x =$ the third part.

Then, $nx =$ the second part,

and $n^2x =$ the first part.

By the conditions,

$$n^2x + nx + x = a$$
$$x(n^2 + n + 1) = a.$$

Whence, $x = \frac{a}{n^2+n+1}$, the third part.

Then, $nx = \frac{an}{n^2+n+1}$, the second part.

and $n^2x = \frac{an^2}{n^2+n+1}$, the third part.

33. Let $x =$ the number of days required.

Then, $\frac{1}{x} =$ what both can do in one day.

Also, $\frac{1}{15} =$ what A can do in one day,

and $\frac{1}{18} =$ what B can do in one day.

By the conditions, $\frac{1}{15} + \frac{1}{18} = \frac{1}{x}$

$$6x + 5x = 90$$
$$11x = 90.$$

Whence, $x = 8\frac{2}{11}$, the number of days required.

34. Let $x =$ the number of hours required.

Then, $\frac{1}{x} =$ what all together can do in one hour.

Also, $\frac{3}{11} =$ what A can do in one hour,

$\frac{4}{11} =$ what B can do in one hour

and $\qquad \frac{5}{11} =$ what C can do in one hour.

By the conditions,

$$\frac{3}{11} + \frac{4}{11} + \frac{5}{11} = \frac{1}{x}$$

$$\frac{12}{11} = \frac{1}{x}.$$

Whence, $\qquad x = \frac{11}{12}$, the number of hours required.

35. Let $\qquad x =$ the side of the court in yards.

Then, $\qquad x + 6 =$ the length of the rectangular area,

and $\qquad x - 4 =$ its breadth.

By the conditions,

$$(x + 6)(x - 4) = x^2$$

$$x^2 + 2x - 24 = x^2$$

$$2x = 24.$$

Whence, $\qquad x = 12$, the side of the court.

Therefore the area of the court is 144 square yards.

36. Let $\qquad x =$ the sum found.

Then, $\qquad \frac{x}{2} - 15 =$ what A received,

and $\qquad \frac{x}{4} + 13 =$ what B received.

By the conditions, $\qquad x = \frac{x}{2} - 15 + \frac{x}{4} + 13 + 27$

$$x = \frac{x}{2} + \frac{x}{4} + 25$$

$$4x = 2x + x + 100.$$

Whence, $\qquad x = 100$, the sum found.

Therefore, $\qquad \frac{x}{2} - 15 = 35$, what A received,

and $\qquad \frac{x}{4} + 13 = 38$, what B received.

37. Let $\qquad x =$ the number of hours required.

Then, $\qquad \frac{1}{x} =$ what both can do in one hour.

Also, $\qquad \frac{1}{a} =$ what A can do in one hour,

and $\qquad \frac{1}{b} =$ what B can do in one hour.

By the conditions, $\frac{1}{a}+\frac{1}{b}=\frac{1}{x}$

$bx+ax=ab.$

Whence, $x=\frac{ab}{a+b}$, the number of hours required.

38. Let $x=$ the number of minutes required.

Then, $\frac{1}{x}=$ what all together can fill in one minute.

Also, $\frac{1}{a}=$ what the first tap can fill in one minute,

$\frac{1}{b}=$ what the second tap can fill in one minute,

and $\frac{1}{c}=$ what the third tap can fill in one minute.

By the conditions,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}$$

$$bcx+acx+abx=abc.$$

Whence, $x=\frac{abc}{ab+bc+ca}$, the number of minutes required.

40. Let $x=$ the number of dollars.

Then, $2x=$ the number of cents,

and $5\times 2x=10x=$ the number of dimes.

Also, $100x=$ the value of x dollars in cents,

and $100x=$ the value of $10x$ dimes in cents.

By the conditions,

$$100x+100x+2x=404$$

$$202x=404.$$

Whence, $x=2$, the number of dollars.

Then, $2x=4$, the number of cents,

and $10x=20$, the number of dimes.

41. Let $x=$ the number of two-penny pieces.

Then, $x+19=$ the number of farthings.

Also, $8x=$ the value of x two-penny pieces in farthings.

By the conditions,

$$8x+x+19=172$$

$$9x=153$$

Whence, $x=17$, the number of two-penny pieces,

and $x+19=36$, the number of farthings.

42. Let $x =$ the cost of the picture in cents.
Then, $x =$ the cost of the frame in cents.
By the conditions,

$$x - 100 = \frac{x + 75}{2}$$

$$2x - 200 = x + 75.$$

Whence, $x = 275$, the cost of the picture in cents.

43. Let $x =$ the number of days he worked.
Then, $36 - x =$ the number of days he was absent.
Also, $\frac{5x}{4} =$ the pay in dollars for x days of work,
and $\frac{36 - x}{2} =$ the deduction for $36 - x$ days of absence.
By the conditions,

$$\frac{5x}{4} - \frac{36 - x}{2} = 17$$

$$5x - 72 + 2x = 68$$

$$7x = 140.$$

Whence, $x = 20$, the number of days he worked,
and $36 - x = 16$, the number of days he was absent.

44. Let $x =$ the sum given to B.
After giving B x dollars, A has $105 - x$ dollars, while B has $83 + x$ dollars. Then by the conditions,

$$105 - x = \frac{83 + x}{3}$$

$$315 - 3x = 83 + x$$

$$4x = 232.$$

Whence, $x = 58$, the sum given to B.

45. Let $x =$ the sum given to B.
Then by the conditions,

$$a - x = c(b + x)$$

$$a - x = bc + cx$$

$$cx + x = a - bc.$$

Whence, $x = \frac{a - bc}{c + 1}$, the sum given to B.

46. Let $x =$ the number of minutes required.
Then, $\frac{1}{x} =$ what all together can do in one minute.

Also, $\frac{1}{80}$ = what the first tap can do in one minute,

$\frac{1}{200}$ = what the second tap can do in one minute,

and $\frac{1}{300}$ = what the third tap can do in one minute.

By the conditions,

$$\frac{1}{80}+\frac{1}{200}+\frac{1}{300}=\frac{1}{x}$$

$$15x+6x+4x=1200$$

$$25x=1200.$$

Whence, $x=48$, the number of minutes required.

48. Let x = the first digit.

Then, $x-4$ = the second digit,

and $2x-4$ = the sum of the digits.

Also, $10x+x-4$,

or $11x-4$ = the number.

By the conditions,

$$\frac{11x-4}{2x-4}=7$$

$$11x-4=14x-28$$

$$3x=24.$$

Whence, $x=8$.

Therefore, $11x-4=84$, the number required.

49. Let x = the second digit.

Then, $3x$ = the first digit,

and $3x-x$, or $2x$ = the difference of the digits.

Also, $30x+x$, or $31x$ = the number.

By the conditions,

$$\frac{31x+3}{2x}=16$$

$$31x+3=32x.$$

Whence, $x=3$.

Therefore, $31x=93$, the number required.

50. Let x = the number of bushels at 9 shillings.

Then, $40-x$ = the number at 13 shillings.

Therefore, $9x$ = the cost in shillings of the first kind,

and $13(40-x)$ = the cost in shillings of the second kind.

By the conditions,

$$9x + 13(40 - x) = 10 \times 40$$
$$9x + 520 - 13x = 400$$
$$4x = 120.$$

Whence, $x = 30$, the number of bushels at 9 shillings,

and $40 - x = 10$, the number at 13 shillings.

51. Let $x =$ the number of ounces of gold.

Then, $4160 - x =$ the number of ounces of silver.

Therefore,

$\frac{x}{19\frac{1}{4}}$, or $\frac{4x}{77} =$ the no. of oz. of water displaced by the gold,

and

$\frac{4160 - x}{10\frac{1}{2}}$, or $\frac{8320 - 2x}{21} =$ the number of ounces displaced by the silver.

By the conditions,

$$\frac{4x}{77} + \frac{8320 - 2x}{21} = 250$$
$$12x + 91520 - 22x = 57750$$
$$10x = 33770.$$

Whence, $x = 3377$, the number of ounces of gold,

and $4160 - x = 783$, the number of ounces of silver.

52. Let $x =$ the first digit.

Then, $x + 3 =$ the second,

and $2x + 3 =$ the sum of the digits.

Also, $10x + x + 3$,

or $11x + 3 =$ the number.

By the conditions,

$$\frac{11x + 3 - 9}{2x + 3} = 3$$
$$11x - 6 = 6x + 9$$
$$5x = 15.$$

Whence, $x = 3$.

Therefore, $11x + 3 = 36$, the number required.

54. Let $x =$ the number of miles.

Then, $\frac{x}{5} =$ the number of hours going,

and $\frac{x}{3\frac{1}{2}}$, or $\frac{2x}{7} =$ the number returning.

By the conditions,

$$\frac{x}{5}+\frac{2x}{7}=\frac{17}{4}$$

$$28x+40x=595$$

$$68x=595.$$

Whence, $x=\frac{595}{68}=8\frac{3}{4}$, the number of miles required.

55. Let $x=$ the no. of days.

Then, $x+n=$ the no. of days travelled by the first courier.

Also, $a(x+n)=$ the no. of miles travelled by the first courier,

and $bx=$ the no. travelled by the second.

By the conditions,

$$bx=a(x+n)$$

$$bx=ax+an$$

$$bx-ax=an.$$

Whence, $x=\frac{an}{b-a}$, the no. of days required.

56. Let $x=$ the number of miles A travels.

Then, $26-x=$ the number B travels.

Also, $\frac{x}{3}=$ the number of hours A takes to travel x miles,

and $\frac{26-x}{4}=$ the number B takes to travel $26-x$ miles.

By the conditions, $\frac{x}{3}=\frac{26-x}{4}+\frac{1}{2}$

$$4x=78-3x+6$$

$$7x=84.$$

Whence, $x=12$, the number of miles A travels,

and $26-x=14$, the number B travels.

57. Let $x=$ the amount invested.

Then, $\frac{3x}{8}=$ the amount in 5 per cent bonds,

and $\frac{5x}{8}=$ the amount in 6 per cent bonds.

Also,

$\frac{5}{100}$ of $\frac{3x}{8}$, or $\frac{3x}{160}=$ the income from the 5 per cents,

and

$\frac{6}{100}$ of $\frac{5x}{8}$, or $\frac{6x}{160}=$ the income from the 6 per cents.

By the conditions,

$$\frac{3x}{160}+\frac{6x}{160}=180$$

$$\frac{9x}{160}=180$$

$$\frac{x}{160}=20.$$

Whence, $x=3200.$

Therefore, $\frac{3x}{8}=1200$, the amount in 5 per cent bonds,

and $\frac{5x}{.8}=2000$, the amount in 6 per cent bonds.

58. Let $x=$ the principal.

By the conditions,

$$\frac{rtx}{100}+x=a$$

$$rtx+100x=100a.$$

Whence, $x=\frac{100a}{rt+100}$, the principal required.

59. Let $x=$ the number of years.

By the conditions,

$$p+\frac{prx}{100}=a$$

$$\frac{prx}{100}=a-p$$

$$prx=100(a-p).$$

Whence, $x=\frac{100(a-p)}{pr}$, the no. of years required.

61. Let $x=$ the divisor.

Then, $37-x=$ the dividend.

By the conditions,

$$\frac{37-x}{x}=3+\frac{1}{x}$$

$$37-x=3x+1$$

$$4x=36.$$

Whence, $x=9$, the divisor,

and $37-x=28$, the dividend.

62. Let $x =$ the divisor.

Then, $113 - x =$ the dividend.

By the conditions,

$$\frac{113 - x}{x} = 2 + \frac{20}{x}$$

$$113 - x = 2x + 20$$

$$3x = 93.$$

Whence, $x = 31$, the divisor,

and $113 - x = 82$, the dividend.

63. Let $x =$ the number on a side at first.

By the conditions,

$$x^2 + 21 = (x+1)^2 - 200$$

$$x^2 + 21 = x^2 + 2x + 1 - 200$$

$$2x = 220.$$

Whence, $x = 110$, the number on a side at first,

and $x^2 + 21 = 12121$, the whole number of troops.

64. Let $x =$ the divisor.

Then, $a - x =$ the dividend.

By the conditions, $\frac{a - x}{x} = b + \frac{c}{x}$

$$a - x = bx + c$$

$$bx + x = a - c.$$

Whence, $x = \frac{a - c}{b + 1}$, the divisor,

and $a - x = a - \frac{a - c}{b + 1} = \frac{ab + c}{b + 1}$, the dividend.

65. Let $x =$ the denominator.

Then, $x - 6 =$ the numerator.

By the conditions, $\frac{x - 6}{x + 8} = \frac{1}{3}$

$$3x - 18 = x + 8$$

$$2x = 26.$$

Whence, $x = 13$, the denominator,

and $x - 6 = 7$, the numerator.

Therefore the fraction is $\frac{7}{13}$.

66. Let $x =$ the first digit.

Then, $6 - x =$ the second.

Also,

$10x + 6 - x$, or $9x + 6 =$ the number.

By the conditions, $9x + 6 = x + 46$

$$8x = 40.$$

Whence, $x = 5$.

Therefore, $9x + 6 = 51$, the number required.

67. Let $x =$ the rate per cent.

By the conditions,

$$p + \frac{ptx}{100} = a$$

$$\frac{ptx}{100} = a - p$$

$$ptx = 100(a - p).$$

Whence, $x = \dfrac{100(a-p)}{pt}$, the rate required.

68. Let $x =$ the price of the picture in cents.

Then, $\dfrac{3x}{4} =$ the price of the frame.

By the conditions,

$$\frac{3x}{4} - 200 = \frac{x + 60}{3}$$

$$9x - 2400 = 4x + 240$$

$$5x = 2640.$$

Whence, $x = 528$, the price of the picture in cents,

and $\dfrac{3x}{4} = 396$, the price of the frame.

69. Let $x =$ the denominator.

Then, $x - 1 =$ the numerator.

By the conditions, $\dfrac{x-1}{x+2} = 2\left(\dfrac{x-1}{x}\right) - 1$

$$x(x-1) = 2(x-1)(x+2) - x(x+2)$$

$$x^2 - x = 2x^2 + 2x - 4 - x^2 - 2x$$

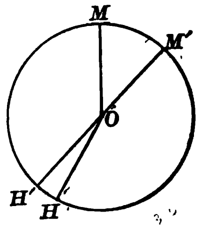
$$-x = -4.$$

Whence, $x = 4$, the denominator,

and $x - 1 = 3$, the numerator.

Therefore the fraction is $\dfrac{3}{4}$.

71. Let OM, OH be the positions of the minute and hour-hands at 7 o'clock, and OM', OH' their positions when opposite to each other.



Let $x =$ the number of minute-spaces in MM'.

Then since the hour-hand travels one-twelfth as fast as the minute-hand,

$\frac{x}{12} =$ the number in HH'.

Also, $35 =$ the number of minute-spaces in $MM'H$,

and $30 =$ the number in $M'HH'$.

Now, $MM' + M'HH' = MM'H + HH'$.

That is, $x + 30 = 35 + \frac{x}{12}$

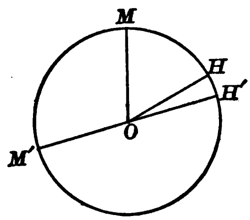
$$\frac{11x}{12} = 5$$

$$11x = 60.$$

Whence, $x = 5\frac{5}{11}$.

Therefore the required time is $5\frac{5}{11}$ minutes after 7 o'clock.

72. Let OM, OH be the positions of the minute and hour-hands at 2 o'clock, and OM', OH' their positions when opposite to each other.



Let $x =$ the number of minute-spaces in $MH'M'$.

Then since the hour-hand travels one-twelfth as fast as the minute-hand,

$\frac{x}{12} =$ the number in HH'.

Also, $10 =$ the number of minute-spaces in MH,

and $30 =$ the number in $H'M'$.

Now, $MH'M' = MH + HH' + H'M'$.

That is, $x = 10 + \frac{x}{12} + 30$

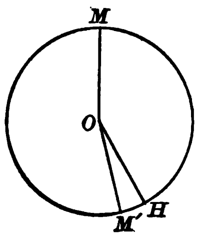
$$\frac{11x}{12} = 40$$

$$11x = 480.$$

Whence, $x = 43\frac{7}{11}$.

Therefore the required time is $43\frac{7}{11}$ minutes after 2 o'clock.

73. Let OM, OH be the positions of the minute and hour-hands at 5 o'clock, and OM' their position when together.



Let $x =$ the number of minute-spaces in MM'.

Then, $\frac{x}{12} =$ the number in HM'.

Also, $25 =$ the number of minute-spaces in MH.

Now, $MM' = MH + HM'$.

That is, $x = 25 + \frac{x}{12}$

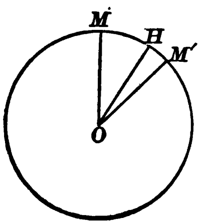
$$\frac{11x}{12} = 25.$$

$$11x = 300.$$

Whence, $x = 27\frac{3}{11}$.

Therefore the required time is $27\frac{3}{11}$ minutes after 5 o'clock.

74. Let OM, OH be the positions of the minute and hour-hands at 1 o'clock, and OM' their position when together.



Let $x =$ the number of minute-spaces in MM'.

Then, $\frac{x}{12} =$ the number in HM'.

Also, $5 =$ the number of minute-spaces in MH.

Now, $MM' = MH + HM'$.

That is, $x = 5 + \frac{x}{12}$

$$\frac{11x}{12} = 5$$

$$11x = 60.$$

Whence, $x = 5\frac{5}{11}$.

Therefore the required time is $5\frac{5}{11}$ minutes after 1 o'clock.

75. Let $x =$ the number of eggs at first.

Then, $\frac{x}{2} + \frac{1}{2} =$ the first sale,

and

$$x - \left(\frac{x}{2} + \frac{1}{2}\right), \text{ or } \frac{x}{2} - \frac{1}{2} = \text{the number remaining.}$$

Therefore,

$$\frac{x}{4}-\frac{1}{4}+\frac{1}{2}, \text{ or } \frac{x}{4}+\frac{1}{4}=\text{the second sale,}$$

and

$$\frac{x}{2}-\frac{1}{2}-\left(\frac{x}{4}+\frac{1}{4}\right), \text{ or } \frac{x}{4}-\frac{3}{4}=\text{the number remaining.}$$

Finally,

$$\frac{x}{8}-\frac{3}{8}+\frac{1}{2}, \text{ or } \frac{x}{8}+\frac{1}{8}=\text{the third sale,}$$

and

$$\frac{x}{4}-\frac{3}{4}-\left(\frac{x}{8}+\frac{1}{8}\right), \text{ or } \frac{x}{8}-\frac{7}{8}=\text{the number remaining.}$$

By the conditions, $\frac{x}{8}-\frac{7}{8}=0$

$$x-7=0.$$

Whence, $x=7$, the number of eggs at first.

76. Let $x=$ the number of dimes.

Then, $20-x=$ the number of half-dimes.

Also, $2x=$ the value of x dimes in half-dimes,

and $27=$ the value of \$1.35 in half-dimes.

By the conditions,

$$2x+20-x=27.$$

Whence, $x=7$, the number of dimes,

and $20-x=13$, the number of half-dimes.

77. Let $x=$ the number of miles.

Then, $\frac{x}{b}=$ the number of hours going,

and $\frac{x}{c}=$ the number returning.

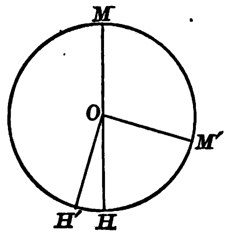
By the conditions,

$$\frac{x}{b}+\frac{x}{c}=a$$

$$cx+bx=abc.$$

Whence, $x=\frac{abc}{b+c}$, the number of miles required.

78. Let OM, OH be the positions of the minute and hour-hands at 6 o'clock, and OM', OH' their positions when at right angles.



Let $x =$ the number of minute-spaces in MM'.

Then, $\frac{x}{12} =$ the number in HH'.

Also, $30 =$ the number of minute-spaces in $MM'H$,

and $15 =$ the number in $M'HH'$.

Now,

$$MM' + M'HH' = MM'H + HH'.$$

That is, $x + 15 = 30 + \frac{x}{12}$

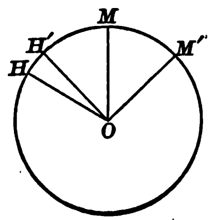
$$\frac{11x}{12} = 15$$

$$11x = 180.$$

Whence, $x = 16\frac{4}{11}$.

Therefore the required time is $16\frac{4}{11}$ minutes after 6 o'clock.

79. The minute and hour-hands will be at right angles to each other once between 10 and 10.30 o'clock, and once between 10.30 and 11 o'clock.



First. Let OM, OH be their positions at 10 o'clock, and OM', OH' their positions when at right angles.

Let $x =$ the number of minute-spaces in MM'.

Then, $\frac{x}{12} =$ the number in HH'.

Also, $10 =$ the number of minute-spaces in $HH'M$,

and $15 =$ the number in $H'MM'$.

Now,

$$HH'M + MM' = HH' + H'MM'.$$

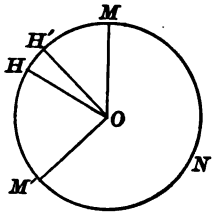
That is, $10 + x = \frac{x}{12} + 15$

$$\frac{11x}{12} = 5$$

$$11x = 60.$$

Whence, $x = 5\frac{5}{11}$.

Second. Let OM, OH be the positions of the minute and hour-hands at 10 o'clock, and OM', OH' their positions when at right angles.



Let $x =$ the number of minute-spaces in MNM'.

Then, $\frac{x}{12} =$ the number in HH'.

Also, $50 =$ the number of minute-spaces in $MNM'H$,

and $15 =$ the number in $M'HH'$.

Now,

$$MNM' + M'HH' = MNM'H + HH'.$$

That is,

$$x + 15 = 50 + \frac{x}{12}$$

$$\frac{11x}{12} = 35$$

$$11x = 420.$$

Whence,

$$x = 38\tfrac{2}{11}$$

Therefore the times are $5\frac{5}{11}$ minutes, or $38\frac{2}{11}$ minutes, after 10 o'clock.

80. Let $x =$ the no. of pieces at a to the dollar.

Then, $c - x =$ the no. at b to the dollar.

Therefore, $\frac{x}{a} =$ the value in dollars of the pieces of the first kind,

and $\frac{c-x}{b} =$ the value of the pieces of the second kind.

By the conditions,

$$\frac{x}{a} + \frac{c-x}{b} = 1$$

$$bx + ac - ax = ab$$

$$ax - bx = ac - ab.$$

Whence, $x = \frac{a(c-b)}{a-b}$, the no. of pieces of the first kind.

Then,

$$c - x = c - \frac{a(c-b)}{a-b}$$

$$= \frac{ac - bc - ac + ab}{a-b}$$

$$= \frac{b(a-c)}{a-b}, \text{ the no. of pieces of the second kind.}$$

81. A performs $\frac{1}{12}$ of the work in an hour, and A and C together perform $\frac{1}{5}$. Hence C alone will perform $\frac{1}{5}-\frac{1}{12}$, or $\frac{7}{60}$ of the work in an hour. Also, B's work is $\frac{3}{2}$ of C's; so that B performs $\frac{3}{2}$ of $\frac{7}{60}$, or $\frac{7}{40}$ of the work in an hour.

Let $\qquad x =$ the no. of hours A works before he is relieved.

Then, $\qquad 7 - x =$ the no. of hours B and C work.

In x hours, A performs $\frac{x}{12}$ of the work; and in $7-x$ hours, B performs $\frac{7(7-x)}{40}$ of the work, and C $\frac{7(7-x)}{60}$ of the work.

Then by the conditions,

$$\frac{x}{12}+\frac{7(7-x)}{40}+\frac{7(7-x)}{60}=1$$

$$10x+21(7-x)+14(7-x)=120$$

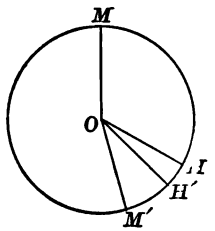
$$10x+147-21x+98-14x=120$$

$$25x=125.$$

Whence, $\qquad x = 5.$

Therefore B and C must relieve A after 5 hours; that is, at 10 A.M.

82. Let OM, OH be the positions of the minute and hour-hands at 4 o'clock, and OM', OH' their positions when the former is five minutes in advance.



Let $\qquad x =$ the number of minute-spaces in $MHH'M'$.

Then, $\qquad \frac{x}{12} =$ the number in HH'.

Also, $\qquad 20 =$ the number of minute-spaces in MH,

and $\qquad 5 =$ the number in $H'M'$.

Now, $\qquad MHH'M' = MH + HH' + H'M'$.

That is, $\qquad x = 20 + \frac{x}{12} + 5$

$$\frac{11x}{12}=25$$

$$11x = 300.$$

Whence, $\qquad x = 27\tfrac{3}{11}.$

Therefore the required time is $27\frac{3}{11}$ minutes after 4 o'clock.

83. Let $x =$ the number bought.

Then, $\frac{10x}{3} =$ the cost of the eggs in cents.

Also, $\frac{7}{2} \times \frac{x}{3}$, or $\frac{7x}{6} =$ the amount received for one-third,

and $\frac{15}{4} \times \frac{2x}{3}$, or $\frac{5x}{2} =$ the amount received for the remainder.

By the conditions,

$$\frac{7x}{6} + \frac{5x}{2} = \frac{10x}{3} + 16$$

$$7x + 15x = 20x + 96$$

$$2x = 96.$$

Whence, $x = 48$, the number of eggs bought.

84. Let $x =$ his capital at first.

Then, $\frac{4x}{3} =$ his capital at the end of the first year,

and $\frac{4x}{3} - 2700 =$ his capital at the beginning of the second year.

Therefore, $\frac{4}{3} \times \left(\frac{4x}{3} - 2700\right)$, or $\frac{16x}{9} - 3600$

$=$ his capital at the end of the second year,

and $\frac{16x}{9} - 3600 - 2700$, or $\frac{16x}{9} - 6300$

$=$ his capital at the beginning of the third year.

Therefore, $\frac{4}{3} \times \left(\frac{16x}{9} - 6300\right)$, or $\frac{64x}{27} - 8400$

$=$ his capital at the end of the third year,

and $\frac{64x}{27} - 8400 - 2700$, or $\frac{64x}{27} - 11100$

$=$ his capital at the beginning of the fourth year.

Therefore, $\frac{4}{3} \times \left(\frac{64x}{27} - 11100\right)$, or $\frac{256x}{81} - 14800$

$=$ his capital at the end of the fourth year.

By the conditions,

$$\frac{256x}{81} - 14800 - 2700 = 2980$$

$$\frac{256x}{81} = 20480$$

$$\frac{x}{81} = 80.$$

Whence, $x = 6480$, the capital at first.

85. Let $x =$ the value of the horse.

The harness and carriage being together worth two-thirds the value of the horse, we have \$25 + the value of the carriage equal to $\frac{2x}{3}$; that is, the value of the carriage is $\frac{2x}{3} - 25$.

Then by the conditions,

$$x + 25 = 2\left(\frac{2x}{3} - 25\right) + 15$$

$$3x + 75 = 4x - 150 + 45.$$

Whence, $x = 180$, the value of the horse,

and $\frac{2x}{3} - 25 = 95$, the value of the carriage.

86. Let $x =$ the no. of miles A travels.

Then, $107 - x =$ the no. B travels.

Also, $\dfrac{x}{\frac{13}{5}}$, or $\frac{5x}{13} =$ the time A takes to travel x miles,

and

$\dfrac{107 - x}{\frac{11}{4}}$, or $\frac{428 - 4x}{11} =$ the time B takes to travel $107 - x$ miles.

By the conditions, $\frac{5x}{13} = \frac{428 - 4x}{11}$

$$55x = 5564 - 52x$$

$$107x = 5564.$$

Whence, $x = 52$, the no. of miles A travels,

and $107 - x = 55$, the no. B travels.

87. Let $x =$ the cost per pound for the mixture.

By the conditions,

$$x(a + b + c) = am + bn + cp.$$

Whence, $x = \frac{am + bn + cp}{a + b + c}$, the cost required.

88. Let $x =$ the no. of days required by A.

Then, $\frac{1}{x} =$ the part A does in one day.

Also,

$\frac{1}{2}$ of $\frac{1}{x}$, or $\frac{1}{2x} =$ the part B does in one day,

and

$\frac{2}{3}$ of $\frac{1}{2x}$, or $\frac{1}{3x} =$ the part C does in one day.

Also, $\frac{1}{6}$ = the part all together do in one day.

By the conditions,

$$\frac{1}{6}=\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x}$$

$$x=6+3+2.$$

Whence, $x=11$, the no. of days required by A.

Therefore, $2x=22$, the no. of days required by B,

and $3x=33$, the no. of days required by C.

89. Let x = B's capital at first.

Then, $\frac{3x}{2}$ = A's capital at first.

Then, $\frac{3x}{2}+150$ = A's capital at the end of the first year.

and $\frac{3x}{4}$ = B's capital at the end of the first year.

Therefore, $\frac{3}{4}\times\left(\frac{3x}{2}+150\right)$, or $\frac{9x}{8}+\frac{225}{2}$

= A's capital at the end of the second year,

and $\frac{3x}{4}+300$ = B's capital at the end of the second year.

By the conditions,

$$\frac{9x}{8}+\frac{225}{2}=\frac{3x}{4}+300$$

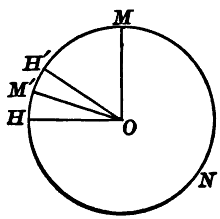
$$9x+900=6x+2400$$

$$3x=1500.$$

Whence, $x=500$, B's capital at first.

Therefore, $\frac{3x}{2}=750$, A's capital at first.

90. Let OM, OH be the positions of the minute and hour-hands at 9 o'clock, and OM', OH' their positions when the latter is one minute in advance.



Let x = the number of minute-spaces in $MNHM'$.

Then, $\frac{x}{12}$ = the number in $HM'H'$.

Also, 45 = the number of minute-spaces in MNH,

and, 1 = the number in $M'H'$.

Now,

$$MNHM'+M'H'=MNH+HM'H.$$

That is, $x + 1 = 45 + \frac{x}{12}$

$$\frac{11x}{12} = 44$$

$$\frac{x}{12} = 4$$

Whence, $x = 48.$

Therefore the required time is 48 minutes after 9 o'clock.

91. Let $x =$ the number of days required by A.

Then, $\frac{1}{x} =$ the part A does in one day.

But, $\frac{7}{12} =$ what A and B together do in one day;

hence, $\frac{7}{12} - \frac{1}{x} =$ the part B does in one day.

Also, $\frac{8}{15} =$ what A and C together do in one day;

hence, $\frac{8}{15} - \frac{1}{x} =$ the part C does in one day.

But, $\frac{9}{20} =$ what B and C together do in one day.

Therefore by the conditions,

$$\frac{7}{12} - \frac{1}{x} + \frac{8}{15} - \frac{1}{x} = \frac{9}{20}$$

$$35x - 60 + 32x - 60 = 27x$$

$$40x = 120.$$

Whence, $x = 3$, the number of days required by A.

Therefore as B does $\frac{7}{12} - \frac{1}{3}$, or $\frac{1}{4}$ in one day, he will require 4 days to complete the work; and as C does $\frac{8}{15} - \frac{1}{3}$, or $\frac{1}{5}$ in one day, he will require 5 days to complete the work.

92. Let $x =$ the number of yards in the smaller piece.

Then, $x + 3 =$ the number in the larger piece.

Therefore, $\frac{7x}{5} =$ the cost of the smaller piece,

and $\frac{5}{6}(x + 3) =$ the cost of the larger.

Also, $\frac{4}{3}(2x + 3) =$ the amount received for the whole.

By the conditions,

$$\frac{4}{3}(2x+3)=\frac{7x}{5}+\frac{5}{6}(x+3)+8$$

$$80x+120=42x+25x+75+240$$

$$13x=195$$

Whence, $x=15$, the no. of yards in the smaller piece,

and $x+3=18$, the no. in the larger.

93. Let $x=$ the number of beggars.

Then, $ax-b=$ the amount distributed,

and $cx+d=$ the same.

By the conditions,

$$ax-b=cx+d$$

$$ax-cx=b+d.$$

Whence, $x=\dfrac{b+d}{a-c}$, the number of beggars.

94. Let $x=$ the amount let.

Then, $\dfrac{21x}{20}=$ the amount due at the end of the first year,

$\dfrac{21}{20}\times\dfrac{21x}{20}$, or $\dfrac{441x}{400}=$ the amount due at the end of the second year,

and

$\dfrac{21}{20}\times\dfrac{441x}{400}$, or $\dfrac{9261x}{8000}=$ the amount due at the end of the third year.

By the conditions,

$$\frac{9261x}{8000}=2315.25$$

$$\frac{x}{8000}=.25$$

Whence, $x=2000$, the amount let.

95. Let $x=$ his annual expenses.

Then, $5000-x$

$=$ his capital at the beginning of the second year.

Therefore, $\dfrac{5}{4}\times(5000-x)$, or $6250-\dfrac{5x}{4}$

$=$ his capital at the end of the second year,

and $6250-\dfrac{5x}{4}-x$, or $6250-\dfrac{9x}{4}$

$=$ his capital at the beginning of the third year.

Therefore, $\dfrac{5}{4}\times\left(6250-\dfrac{9x}{4}\right)$, or $\dfrac{15625}{2}-\dfrac{45x}{16}$

$=$ his capital at the end of the third year.

By the conditions,

$$\frac{15625}{2} - \frac{45x}{16} - x = 2475$$

$$125000 - 45x - 16x = 39600$$

$$61x = 85400.$$

Whence, $x = 1400$, the annual expenses

96. Let $x =$ the amount of his property.

Then, $\frac{x}{3} =$ the amount invested in $3\frac{1}{2}$ per cents,

$\frac{2x}{5} =$ the amount invested in 4 per cents,

and $\frac{4x}{15} =$ the amount invested in $4\frac{1}{2}$ per cents.

Therefore,

$\frac{7}{200}$ of $\frac{x}{3}$, or $\frac{7x}{600} =$ the income from the $3\frac{1}{2}$ per cents,

$\frac{4}{100}$ of $\frac{2x}{5}$, or $\frac{8x}{500} =$ the income from the 4 per cents,

and

$\frac{9}{200}$ of $\frac{4x}{15}$, or $\frac{36x}{3000} =$ the income from the $4\frac{1}{2}$ per cents.

By the conditions,

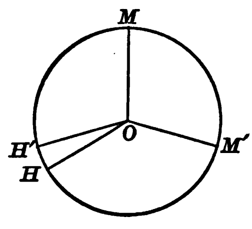
$$\frac{7x}{600} + \frac{8x}{500} + \frac{36x}{3000} = 595$$

$$35x + 48x + 36x = 1785000$$

$$119x = 1785000$$

Whence, $x = 15000$, the amount of his property.

97. Let OM, OH be the positions of the minute and hour hands at 8 o'clock, and OM', OH' their positions when the former is thirty-five minutes in advance.



Let $x =$ the number of minute-spaces in MM'.

Then, $\frac{x}{12} =$ the number in HH'.

Also, $20 =$ the number of minute-spaces in $HH'M$,

and $35 =$ the number in $H'MM'$.

Now,

$$HH'M + MM' = HH' + H'MM'.$$

That is,

$$20 + x = \frac{x}{12} + 35$$

$$\frac{11x}{12} = 15$$

$$11x = 180.$$

Whence, $x = 16\tfrac{4}{11}$.

Therefore the required time is $16\tfrac{4}{11}$ minutes after 8 o'clock.

98. Let $x =$ the number of leaps made by the fox.

Then, $\frac{2x}{3} =$ the number made by the greyhound.

Now one leap of the greyhound is equivalent to $\frac{7}{3}$ leaps of the fox;

hence, $\frac{2x}{3}$ leaps of the greyhound are equivalent to $\frac{2x}{3} \times \frac{7}{3}$, or $\frac{14x}{9}$ leaps of the fox.

But the distance traversed by the greyhound is equal to the distance traversed by the fox, plus 60 leaps of the fox; therefore,

$$\frac{14x}{9} = x + 60$$

$$14x = 9x + 540$$

$$5x = 540$$

Whence, $x = 108$, the number of leaps made by the fox,

and $\frac{2x}{3} = 72$, the number made by the greyhound.

CHAPTER XIV.

Art. 184.—Pages 134, 135.

3. $\begin{cases} 7x + 2y = 31 & (1) \\ 3x - 4y = 23 & (2) \end{cases}$

Multiplying (1) by 2,

$$14x + 4y = 62 \qquad (3)$$

Adding (2) and (3),

$$17x = 85$$
$$x = 5$$

Substituting in (1),

$$35 + 2y = 31$$
$$2y = -4$$
$$y = -2$$

4. $\begin{cases} 3x + 7y = 33 & (1) \\ x + 2y = 10 & (2) \end{cases}$

Multiplying (2) by 3,

$$3x + 6y = 30 \qquad (3)$$

Subtracting (3) from (1),

$$y = 3$$

Substituting in (2),

$$x + 6 = 10$$
$$x = 4$$

5. $\begin{cases} 2x - 3y = 4 & (1) \\ 6x - y = 28 & (2) \end{cases}$

Multiplying (1) by 3,

$$6x - 9y = 12 \qquad (3)$$

Subtracting (3) from (2),

$$8y = 16$$
$$y = 2$$

Substituting in (1),

$$2x - 6 = 4$$
$$2x = 10$$
$$x = 5$$

6. $\begin{cases} 7y - 5x = -11 & (1) \\ 15x - 14y = 82 & (2) \end{cases}$

Multiplying (1) by 2,

$$14y - 10x = -22 \qquad (3)$$

Adding (2) and (3),

$$5x = 60$$
$$x = 12$$

Substituting in (1),

$$7y - 60 = -11$$
$$7y = 49$$
$$y = 7$$

7. $\begin{cases} 2x - 3y = -24 & (1) \\ 3x + 2y = 3 & (2) \end{cases}$

Multiplying (1) by 2, and (2) by 3,

$$4x - 6y = -48$$
$$\underline{9x + 6y = 9}$$

Adding, $13x = -39$

$$x = -3$$

Substituting in (2),

$$-9 + 2y = 3$$
$$2y = 12$$
$$y = 6$$

8. $\begin{cases} 9x - 13y = 76 & (1) \\ 15x + 4y = 101 & (2) \end{cases}$

Multiplying (1) by 5, and (2) by 3,

$$45x - 65y = 380$$
$$\underline{45x + 12y = 303}$$

Subtracting, $-77y = 77$

$$y = -1$$

Substituting in (1),

$$9x + 13 = 76$$
$$9x = 63$$
$$x = 7$$

9. $\begin{cases} 24x + 13y = -27 & (1) \\ 36x + 11y = -15 & (2) \end{cases}$

Multiplying (1) by 3, and (2) by 2,

$$\begin{aligned} 72x + 39y &= -81 \\ 72x + 22y &= -30 \end{aligned}$$

Subtracting, $17y = -51$

$$y = -3$$

Substituting in (1),

$$\begin{aligned} 24x - 39 &= -27 \\ 24x &= 12 \\ x &= \frac{1}{2} \end{aligned}$$

10. $\begin{cases} 15y - 8x = 12 & (1) \\ 25y + 12x = 1 & (2) \end{cases}$

Multiplying (1) by 3, and (2) by 2,

$$\begin{aligned} 45y - 24x &= 36 \\ 50y + 24x &= 2 \end{aligned}$$

Adding, $95y = 38$

$$y = \frac{38}{95} = \frac{2}{5}$$

Substituting in (1),

$$\begin{aligned} 6 - 8x &= 12 \\ -8x &= 6 \\ x &= -\frac{6}{8} = -\frac{3}{4} \end{aligned}$$

11. $\begin{cases} 5x - 7y = 15 & (1) \\ 3x - 5y = 13 & (2) \end{cases}$

Multiplying (1) by 3, and (2) by 5,

$$\begin{aligned} 15x - 21y &= 45 \\ 15x - 25y &= 65 \end{aligned}$$

Subtracting, $4y = -20$

$$y = -5$$

Substituting in (2),

$$\begin{aligned} 3x + 25 &= 13 \\ 3x &= -12 \\ x &= -4 \end{aligned}$$

12. $\begin{cases} 7x - 11y = -58 & (1) \\ 15x + 8y = 2 & (2) \end{cases}$

Multiplying (1) by 8, and (2) by 11,

$$\begin{aligned} 56x - 88y &= -464 \\ 165x + 88y &= 22 \end{aligned}$$

Adding, $221x = -442$

$$x = -2$$

Substituting in (2),

$$\begin{aligned} -30 + 8y &= 2 \\ 8y &= 32 \\ y &= 4 \end{aligned}$$

13. $\begin{cases} 11y - 18x = 2 & (1) \\ 24x - 5y = -22 & (2) \end{cases}$

Multiplying (1) by 4, and (2) by 3,

$$\begin{aligned} 44y - 72x &= 8 \\ 72x - 15y &= -66 \end{aligned}$$

Adding, $29y = -58$

$$y = -2$$

Substituting in (1),

$$\begin{aligned} -22 - 18x &= 2 \\ -18x &= 24 \\ x &= -\frac{4}{3} \end{aligned}$$

14. $\begin{cases} 24x - 18y = -43 & (1) \\ 42x + 30y = 17 & (2) \end{cases}$

Multiplying (1) by 5, and (2) by 3,

$$\begin{aligned} 120x - 90y &= -215 \\ 126x + 90y &= 51 \end{aligned}$$

Adding, $246x = -164$

$$x = -\frac{2}{3}$$

Substituting in (2),

$$\begin{aligned} -28 + 30y &= 17 \\ 30y &= 45 \\ y &= \frac{3}{2} \end{aligned}$$

15. $\begin{cases} 11x - 12y = -32 & (1) \\ 11y - 12x = 14 & (2) \end{cases}$

Multiplying (1) by 11, and (2) by 12,

$$121x - 132y = -352$$
$$132y - 144x = 168$$

Adding, $\quad -23x = -184$

$$x = 8$$

Substituting in (2),

$$11y - 96 = 14$$
$$11y = 110$$
$$y = 10$$

16. $\begin{cases} 9x - 11y = 24 & (1) \\ 10x + 9y = -37 & (2) \end{cases}$

Multiplying (1) by 9, and (2) by 11,

$$81x - 99y = 216$$
$$110x + 99y = -407$$

Adding, $\quad 191x = -191$

$$x = -1$$

Substituting in (2),

$$-10 + 9y = -37$$
$$9y = -27$$
$$y = -3$$

17. $\begin{cases} 12x + 21y = -23 & (1) \\ 15x + 28y = -30 & (2) \end{cases}$

Multiplying (1) by 4, and (2) by 3,

$$48x + 84y = -92$$
$$45x + 84y = -90$$

Subtracting, $\quad 3x = -2$

$$x = -\frac{2}{3}$$

Substituting in (1),

$$-8 + 21y = -23$$
$$21y = -15$$
$$y = -\frac{5}{7}$$

Art. 185. — Page 136.

2. $\begin{cases} x + y = 7 & (1) \\ 3x + 2y = 19 & (2) \end{cases}$

From (1), $\quad y = 7 - x \qquad (3)$

Substituting in (2),

$$3x + 2(7 - x) = 19$$
$$3x + 14 - 2x = 19$$
$$x = 5$$

Substituting in (3),

$$y = 7 - 5 = 2$$

3. $\begin{cases} 3x - y = 10 & (1) \\ x + 4y = -1 & (2) \end{cases}$

From (2), $\quad x = -4y - 1 \qquad (3)$

Substituting in (1),

$$3(-4y - 1) - y = 10$$
$$-12y - 3 - y = 10$$
$$-13y = 13$$
$$y = -1$$

Substituting in (3),

$$x = 4 - 1 = 3$$

4. $\begin{cases} 3x - 4y = 2 & (1) \\ 2x - 5y = 6 & (2) \end{cases}$

From (2), $2x = 5y + 6$

Or, $x = \frac{5y+6}{2}$ (3)

Substituting in (1),

$$3\left(\frac{5y+6}{2}\right) - 4y = 2$$
$$15y + 18 - 8y = 4$$
$$7y = -14$$
$$y = -2$$

Substituting in (3),

$$x = \frac{-10+6}{2} = -2$$

5. $\begin{cases} 7x - 2y = 8 & (1) \\ 8y - 5x = -9 & (2) \end{cases}$

From (1), $7x - 8 = 2y$

Or, $y = \frac{7x-8}{2}$ (3)

Substituting in (2),

$$8\left(\frac{7x-8}{2}\right) - 5x = -9$$
$$28x - 32 - 5x = -9$$
$$23x = 23$$
$$x = 1$$

Substituting in (3),

$$y = \frac{7-8}{2} = -\frac{1}{2}$$

6. $\begin{cases} 9x - 4y = -4 & (1) \\ 15x + 8y = -3 & (2) \end{cases}$

From (1),

$$9x + 4 = 4y$$

Or, $y = \frac{9x+4}{4}$ (3)

Substituting in (2),

$$15x + 8\left(\frac{9x+4}{4}\right) = -3$$
$$15x + 18x + 8 = -3$$
$$33x = -11$$
$$x = -\frac{1}{3}$$

Substituting in (3),

$$y = \frac{-3+4}{4} = \frac{1}{4}$$

7. $\begin{cases} 2x - 7y = 8 & (1) \\ 4y - 9x = 19 & (2) \end{cases}$

From (1), $2x = 7y + 8$

Or, $x = \frac{7y+8}{2}$ (3)

Substituting in (2),

$$4y - 9\left(\frac{7y+8}{2}\right) = 19$$
$$8y - 63y - 72 = 38$$
$$-55y = 110$$
$$y = -2$$

Substituting in (3),

$$x = \frac{-14+8}{2} = -3$$

8. $\begin{cases} 5x + 7y = -19 & (1) \\ 4x + 5y = -14 & (2) \end{cases}$

From (2), $4x = -5y - 14$

Or, $x = -\frac{5y+14}{4}$ (3)

Substituting in (1),

$$-5\left(\frac{5y+14}{4}\right) + 7y = -19$$
$$-25y - 70 + 28y = -76$$
$$3y = -6$$
$$y = -2$$

Substituting in (3),

$$x = -\frac{-10+14}{4} = -1$$

9. $\begin{cases} 10x - 7y = 9 & (1) \\ 4y - 15x = -7 & (2) \end{cases}$

From (2), $4y = 15x - 7$

Or, $y = \dfrac{15x - 7}{4}$ (3)

Substituting in (1),

$$10x - 7\left(\frac{15x-7}{4}\right) = 9$$
$$40x - 105x + 49 = 36$$
$$-65x = -13$$
$$x = \frac{1}{5}$$

Substituting in (3),

$$y = \frac{3-7}{4} = -1$$

10. $\begin{cases} 6x - 5y = -7 & (1) \\ 10x + 3y = 11 & (2) \end{cases}$

From (2), $3y = 11 - 10x$

Or, $y = \dfrac{11 - 10x}{3}$ (3)

Substituting in (1),

$$6x - 5\left(\frac{11-10x}{3}\right) = -7$$
$$18x - 55 + 50x = -21$$
$$68x = 34$$
$$x = \frac{1}{2}$$

Substituting in (3),

$$y = \frac{11-5}{3} = 2$$

11. $\begin{cases} 9x + 2y = 15 & (1) \\ 4x + 7y = 3 & (2) \end{cases}$

From (1), $2y = 15 - 9x$

Or, $y = \dfrac{15 - 9x}{2}$ (3)

Substituting in (2),

$$4x + 7\left(\frac{15-9x}{2}\right) = 3$$
$$8x + 105 - 63x = 6$$
$$-55x = -99$$
$$x = \frac{9}{5}$$

Substituting in (3),

$$y = \frac{15 - \frac{81}{5}}{2} = -\frac{3}{5}$$

12. $\begin{cases} 8x + 7y = -23 & (1) \\ 5y - 12x = -12 & (2) \end{cases}$

From (2),

$$5y + 12 = 12x$$

Or, $x = \dfrac{5y + 12}{12}$ (3)

Substituting in (1),

$$8\left(\frac{5y+12}{12}\right) + 7y = -23$$

Or,

$$2\left(\frac{5y+12}{3}\right) + 7y = -23$$
$$10y + 24 + 21y = -69$$
$$31y = -93$$
$$y = -3$$

Substituting in (3),

$$x = \frac{-15+12}{12} = -\frac{1}{4}$$

13. $\begin{cases} 7y - 3x = 139 & (1) \\ 2x + 5y = 91 & (2) \end{cases}$

From (2), $2x = 91 - 5y$

Or, $x = \dfrac{91 - 5y}{2}$ (3)

Substituting in (1),

$$7y - 3\left(\frac{91-5y}{2}\right) = 139$$
$$14y - 273 + 15y = 278$$
$$29y = 551$$
$$y = 19$$

Substituting in (3),

$$x = \frac{91 - 95}{2} = -2$$

Art. 186.—Page 137.

2. $\begin{cases} x - y = -1 & (1) \\ 3x + 5y = 21 & (2) \end{cases}$

From (1), $x = y - 1 \qquad (3)$

From (2), $3x = 21 - 5y$

Or, $x = \frac{21 - 5y}{3} \qquad (4)$

From (3) and (4),

$$y - 1 = \frac{21 - 5y}{3}$$

$$3y - 3 = 21 - 5y$$

$$8y = 24$$

$$y = 3$$

Substituting in (3),

$$x = 3 - 1 = 2$$

3. $\begin{cases} 6x + 5y = -8 & (1) \\ 4x + 3y = -5 & (2) \end{cases}$

From (1), $6x = -5y - 8$

Or, $x = -\frac{5y + 8}{6} \qquad (3)$

From (2), $4x = -3y - 5$

Or, $x = -\frac{3y + 5}{4} \qquad (4)$

From (3) and (4),

$$\frac{5y + 8}{6} = \frac{3y + 5}{4}$$

$$10y + 16 = 9y + 15$$

$$y = -1$$

Substituting in (4),

$$x = -\frac{-3 + 5}{4} = -\frac{1}{2}$$

4. $\begin{cases} 3x - 5y = 25 & (1) \\ 7y - 2x = -24 & (2) \end{cases}$

From (1), $3x = 5y + 25$

Or, $x = \frac{5y + 25}{3} \qquad (3)$

From (2),

$$7y + 24 = 2x$$

Or, $x = \frac{7y + 24}{2} \qquad (4)$

From (3) and (4),

$$\frac{5y + 25}{3} = \frac{7y + 24}{2}$$

$$10y + 50 = 21y + 72$$

$$-11y = 22$$

$$y = -2$$

Substituting in (3),

$$x = \frac{-10 + 25}{3} = 5$$

5. $\begin{cases} 3x - 10y = -36 & (1) \\ 2x - 9y = -31 & (2) \end{cases}$

From (1), $3x = 10y - 36$

Or, $x = \frac{10y - 36}{3} \qquad (3)$

From (2), $2x = 9y - 31$

Or, $x = \frac{9y - 31}{2} \qquad (4)$

From (3) and (4),

$$\frac{10y - 36}{3} = \frac{9y - 31}{2}$$

$$20y - 72 = 27y - 93$$

$$-7y = -21$$

$$y = 3$$

Substituting in (4),

$$x = \frac{27 - 31}{2} = -2$$

6. $\begin{cases} 3x - 5y = 51 & (1) \\ 2x + 7y = 3 & (2) \end{cases}$

From (1), $3x = 5y + 51$

Or, $x = \frac{5y + 51}{3} \qquad (3)$

From (2), $2x = 3 - 7y$

Or, $x = \frac{3 - 7y}{2} \qquad (4)$

From (3) and (4),

$$\frac{5y+51}{3}=\frac{3-7y}{2}$$

$$10y+102=9-21y$$

$$31y=-93$$

$$y=-3$$

Substituting in (3),

$$x=\frac{-15+51}{3}=12$$

7. $\begin{cases} 7x+y=-3 & (1) \\ x+6y=23 & (2) \end{cases}$

From (1), $y=-7x-3$ (3)

From (2), $6y=23-x$

Or, $y=\frac{23-x}{6}$ (4)

From (3) and (4),

$$-7x-3=\frac{23-x}{6}$$

$$-42x-18=23-x$$

$$-41x=41$$

$$x=-1$$

Substituting in (2),

$$y=7-3=4$$

8. $\begin{cases} 5x+6y=24 & (1) \\ 9y-8x=-26 & (2) \end{cases}$

From (1), $6y=24-5x$

Or, $y=\frac{24-5x}{6}$ (3)

From (2), $9y=8x-26$

Or, $y=\frac{8x-26}{9}$ (4)

From (3) and (4),

$$\frac{24-5x}{6}=\frac{8x-26}{9}$$

$$72-15x=16x+52$$

$$-31x=-124$$

$$x=4$$

Substituting in (3),

$$y=\frac{24-20}{6}=\frac{2}{3}$$

9. $\begin{cases} 7x-8y=-11 & (1) \\ x-12y=12 & (2) \end{cases}$

From (1), $7x=8y-11$

Or, $x=\frac{8y-11}{7}$ (3)

From (2), $x=12y+12$ (4)

From (3) and (4),

$$\frac{8y-11}{7}=12y+12$$

$$8y-11=84y+84$$

$$-76y=95$$

$$y=-\frac{95}{76}=-\frac{5}{4}$$

Substituting in (4),

$$x=-15+12=-3$$

10. $\begin{cases} 5x-12y=7 & (1) \\ 10x-9y=4 & (2) \end{cases}$

From (1), $5x=12y+7$

Or, $x=\frac{12y+7}{5}$ (3)

From (2), $10x=9y+4$

Or, $x=\frac{9y+4}{10}$ (4)

From (3) and (4),

$$\frac{12y+7}{5}=\frac{9y+4}{10}$$

$$24y+14=9y+4$$

$$15y=-10$$

$$y=-\frac{2}{3}$$

Substituting in (4),

$$x=\frac{-6+4}{10}=-\frac{1}{5}$$

11. $\begin{cases} 7y-12x=17 & (1) \\ 8x+11y=20 & (2) \end{cases}$

From (1),

$$7y-17=12x$$

Or, $x=\frac{7y-17}{12}$ (3)

From (2), $8x=20-11y$

Or, $x=\frac{20-11y}{8}$ (4)

From (3) and (4),

$$\frac{7y-17}{12}=\frac{20-11y}{8}$$

$$14y-34=60-33y$$

$$47y=94$$

$$y=2$$

Substituting in (3),

$$x=\frac{14-17}{12}=-\frac{1}{4}$$

12. $\begin{cases}7x+3y=6 & (1)\\ 11x+9y=8 & (2)\end{cases}$

From (1), $3y=6-7x$

Or, $y=\frac{6-7x}{3}$ (3)

From (2), $9y=8-11x$

Or, $y=\frac{8-11x}{9}$ (4)

From (3) and (4),

$$\frac{6-7x}{3}=\frac{8-11x}{9}$$

$$18-21x=8-11x$$

$$-10x=-10$$

$$x=1$$

Substituting in (3),

$$y=\frac{6-7}{3}=-\frac{1}{3}$$

13. $\begin{cases}15x+6y=-7 & (1)\\ 8y-21x=18 & (2)\end{cases}$

From (1), $6y=-15x-7$

Or, $y=-\frac{15x+7}{6}$ (3)

From (2), $8y=21x+18$

Or, $y=\frac{21x+18}{8}$ (4)

From (3) and (4),

$$-\frac{15x+7}{6}=\frac{21x+18}{8}$$

$$-60x-28=63x+54$$

$$-123x=82$$

$$x=-\frac{82}{123}=-\frac{2}{3}$$

Substituting in (4),

$$y=\frac{-14+18}{8}=\frac{1}{2}$$

Art. 187.—Pages 138–143.

2. $\begin{cases}11y+6x=115 & (1)\\ \frac{2x}{3}-\frac{11y}{6}=-\frac{5}{2} & (2)\end{cases}$

From (2),

$4x-11y=-15$ (3)

Adding (1) and (3),

$$10x=100$$

$$x=10$$

Substituting in (1),

$$11y+60=115$$

$$11y=55$$

$$y=5$$

3. $\begin{cases}\frac{x}{3}+3y=-46 & (1)\\ \frac{y}{3}+3x=66 & (2)\end{cases}$

From (1), $x+9y=-138$ (3)

From (2), $y+9x=198$ (4)

Adding (3) and (4),

$$10x+10y=60$$

$x+y=6$ (5)

Subtracting (5) from (4),

$$8x=192$$

$$x=24$$

Subtracting (5) from (3),

$$8y=-144$$

$$y=-18$$

4. $\begin{cases} \dfrac{3x}{2} - \dfrac{5y}{3} = -\ 4 & (1) \\ \dfrac{x}{8} + \dfrac{y}{6} = -\ 4 & (2) \end{cases}$

From (1),

$$9x - 10y = -\ 24 \qquad (3)$$

From (2),

$$3x + \ 4y = -\ 96 \qquad (4)$$

Multiplying (4) by 3,

$$9x + 12y = -288 \qquad (5)$$

Subtracting (3) from (5),

$$22y = -264$$
$$y = -\ 12$$

Substituting in (4),

$$3x - 48 \ = -\ 96$$
$$3x = -\ 48$$
$$x = -\ 16$$

5. $\begin{cases} \dfrac{x}{3} - \dfrac{y}{4} = \dfrac{5}{6} & (1) \\ \dfrac{x}{5} - \dfrac{y}{6} = \dfrac{47}{90} & (2) \end{cases}$

From (1),

$$4x - \ 3y = 10 \qquad (3)$$

From (2),

$$18x - 15y = 47 \qquad (4)$$

Multiplying (3) by 5,

$$20x - 15y = 50 \qquad (5)$$

Subtracting (4) from (5),

$$2x = \ 3$$
$$x = \frac{3}{2}$$

Substituting in (3),

$$6 - 3y = 10$$
$$-3y = \ 4$$
$$y = -\frac{4}{3}$$

6. $\begin{cases} .2x - .05y = .25 & (1) \\ .03x + .3y = .96 & (2) \end{cases}$

Multiplying (1) by .3, and (2) by 2,

$$.06x - .015y = .075 \qquad (3)$$
$$.06x + .6 \ \ y = 1.92 \qquad (4)$$

Subtracting (3) from (4),

$$.615y = 1.845$$
$$y = \frac{1.845}{.615} = 3$$

Substituting in (1),

$$.2x - .15 = .25$$
$$.2x = .4$$
$$x = \frac{.4}{.2} = 2$$

7. $\begin{cases} .5x + \ 2y = .01 & (1) \\ .11x + .3y = -.009 & (2) \end{cases}$

Multiplying (1) by .3, and (2) by 2,

$$.15x + .6y = .003 \qquad (3)$$
$$.22x + .6y = -.018 \qquad (4)$$

Subtracting (3) from (4),

$$.07x = -.021$$
$$x = -\frac{.021}{.07} = -.3$$

Substituting in (1),

$$-.15 + 2y = .01$$
$$2y = .16$$
$$y = .08$$

8. $\begin{cases} \dfrac{x+y}{2} - \dfrac{x-y}{3} = \ 8 & (1) \\ \dfrac{x+y}{3} + \dfrac{x-y}{4} = \ 11 & (2) \end{cases}$

From (1),

$$3x + 3y - 2x + 2y = \ 48$$

Or, $\qquad x + 5y = \ 48 \qquad (3)$

From (2),

$$4x + 4y + 3x - 3y = 132$$

Or, $\qquad 7x + y = 132 \qquad (4)$

Multiplying (4) by 5,

$$35x + 5y = 660$$
$$\underline{x + 5y = \ 48} \qquad (3)$$

Subtracting, $\qquad 34x = 612$

$$x = \ 18$$

Substituting in (4),

$$126 + y = 132$$
$$y = \ 6$$

9. $\begin{cases} 10 - \dfrac{2x+3y}{5} = \dfrac{y}{3} & (1) \\ \dfrac{4y-3x}{6} = \dfrac{3x}{4} + 1 & (2) \end{cases}$

From (1), $150 - 6x - 9y = 5y$

$6x + 14y = 150$

Or, $3x + 7y = 75$ (3)

From (2), $8y - 6x = 9x + 12$

Or, $8y - 15x = 12$ (4)

Multiplying (3) by 5, $15x + 35y = 375$ (5)

Adding (4) and (5), $43y = 387$

$y = 9$

Substituting in (3), $3x + 63 = 75$

$3x = 12$

$x = 4$

10. $\begin{cases} x(2y-3) = 2y(x+1) & (1) \\ \dfrac{3}{x-1} + \dfrac{5}{y+2} = 0 & (2) \end{cases}$

From (1), $2xy - 3x = 2xy + 2y$

Or, $3x + 2y = 0$ (3)

From (2), $3y + 6 + 5x - 5 = 0$

Or, $3y + 5x = -1$ (4)

From (3), $y = -\dfrac{3x}{2}$ (5)

Substituting in (4), $-\dfrac{9x}{2} + 5x = -1$

$-9x + 10x = -2$

$x = -2$

Substituting in (5), $y = 3$

11. $\begin{cases} x(y-3) - y(x+4) = 22 & (1) \\ (y+1)(x-2) - (y+3)(x-4) = 6 & (2) \end{cases}$

From (1), $xy - 3x - xy - 4y = 22$

Or, $3x + 4y = -22$ (3)

From (2), $xy + x - 2y - 2 - xy - 3x + 4y + 12 = 6$

Or, $-2x + 2y = -4$ (4)

Multiplying (4) by 2, $-4x + 4y = -8$ (5)

Subtracting (5) from (3), $7x = -14$

$x = -2$

Substituting in (4), $4 + 2y = -4$

$2y = -8$

$y = -4$

12. $\begin{cases} \dfrac{x+y}{x-y}=\dfrac{5}{3} & (1) \\ \dfrac{x+y+1}{x-y-1}=7 & (2) \end{cases}$

From (1), $3x+3y=5x-5y$

$8y-2x=0$

Or, $4y-x=0$ (3)

From (2), $x+y+1=7x-7y-7$

$8y-6x=-8$

Or, $4y-3x=-4$ (4)

Subtracting (4) from (3), $2x=4$

$x=2$

Substituting in (3), $4y-2=0$

$y=\frac{1}{2}$

13. $\begin{cases} \dfrac{x}{2}-12=\dfrac{y}{4}+8 & (1) \\ \dfrac{x+y}{5}-\dfrac{2y-x}{4}=15 & (2) \end{cases}$

From (1), $2x-48=y+32$

Or, $2x-y=80$ (3)

From (2), $4x+4y-10y+5x=300$

$9x-6y=300$

Or, $3x-2y=100$ (4)

Multiplying (3) by 2, $4x-2y=160$ (5)

Subtracting (4) from (5), $x=60$

Substituting in (3), $120-y=80$

$y=40$

14. $\begin{cases} x-\dfrac{3x+2}{5}=\dfrac{y+2}{3} & (1) \\ y-\dfrac{2y+1}{3}=\dfrac{x-6}{5} & (2) \end{cases}$

From (1), $15x-9x-6=5y+10$

Or, $6x-5y=16$ (3)

From (2), $15y-10y-5=3x-18$

Or, $5y-3x=-13$ (4)

Adding (3) and (4), $3x=3$

$x=1$

Substituting in (4), $5y-3=-13$

$5y=-10$

$y=-2$

15. $$\begin{cases} \dfrac{2x+3y}{x+y+13} = -\dfrac{1}{2} & (1) \\[2ex] \dfrac{5x}{3} - \dfrac{7y-2}{5} = 11 & (2) \end{cases}$$

From (1), $4x + 6y = -x - y - 13$

Or, $5x + 7y = -13$ (3)

From (2), $25x - 21y + 6 = 165$

Or, $25x - 21y = 159$ (4)

Multiplying (3) by 3, $15x + 21y = -39$ (5)

Adding (4) and (5), $40x = 120$

$x = 3$

Substituting in (3), $15 + 7y = -13$

$7y = -28$

$y = -4$

16. $$\begin{cases} \dfrac{3x+7}{6} - \dfrac{7-2y}{10} = x & (1) \\[2ex] \dfrac{2y-3}{6} - \dfrac{5-3x}{8} = y & (2) \end{cases}$$

From (1), $15x + 35 - 21 + 6y = 30x$

Or, $6y - 15x = -14$ (3)

From (2), $8y - 12 - 15 + 9x = 24y$

Or, $9x - 16y = 27$ (4)

Multiplying (3) by 3, and (4) by 5,
$$\begin{aligned} 18y - 45x &= -42 \\ 45x - 80y &= 135 \end{aligned}$$

Adding, $-62y = 93$

$y = -\dfrac{3}{2}$

Substituting in (4), $9x + 24 = 27$

$9x = 3$

$x = \dfrac{1}{3}$

17. $$\begin{cases} \dfrac{x+3y}{2x-y} = -\dfrac{3}{8} & (1) \\[2ex] \dfrac{7y-x}{2+x+2y} = -17 & (2) \end{cases}$$

From (1), $8x + 24y = -6x + 3y$

$14x + 21y = 0$

Or, $2x + 3y = 0$ (3)

From (2), $7y - x = -34 - 17x - 34y$

Or, $41y + 16x = -34$ (4)

Multiplying (3) by 8, $16x + 24y = 0$ (5)

Subtracting (5) from (4),

$$17y = -34$$
$$y = -2$$

Substituting in (3),

$$2x - 6 = 0$$
$$2x = 6$$
$$x = 3$$

18. $$\begin{cases} \dfrac{x-5}{4} - \dfrac{2x-y-1}{3} = \dfrac{2y-2}{5} & (1) \\[2ex] \dfrac{2y+x-1}{9} = \dfrac{x+y}{4} & (2) \end{cases}$$

From (1), $15x - 75 - 40x + 20y + 20 = 24y - 24$

Or, $-25x - 4y = 31$ (3)

From (2), $8y + 4x - 4 = 9x + 9y$

Or, $-5x - y = 4$ (4)

Multiplying (4) by 4,

$$\begin{array}{rr} -20x - 4y = 16 & \\ -25x - 4y = 31 & (3) \\ \hline \end{array}$$

Subtracting,

$$5x = -15$$
$$x = -3$$

Substituting in (4),

$$15 - y = 4$$
$$y = 11$$

19. $$\begin{cases} \dfrac{\frac{3x}{4} - \frac{y}{3}}{\frac{1}{2}} - \dfrac{\frac{x}{2} + \frac{2y}{5}}{\frac{13}{4}} = -\dfrac{7}{6} & (1) \\[3ex] 4y - 3x = 11 & (2) \end{cases}$$

From (1),

$$\frac{3x}{2} - \frac{2y}{3} - \frac{2x}{13} - \frac{8y}{65} = -\frac{7}{6}$$
$$585x - 260y - 60x - 48y = -455$$
$$525x - 308y = -455$$

Or, $75x - 44y = -65$ (3)

Multiplying (2) by 25, $100y - 75x = 275$ (4)

Adding (3) and (4),

$$56y = 210$$
$$y = \frac{15}{4}$$

Substituting in (2),

$$15 - 3x = 11$$
$$3x = 4$$
$$x = \frac{4}{3}$$

20. $\begin{cases} \dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5} & (1) \\ 8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3} & (2) \end{cases}$

From (1), $15x-25y+30=4x+2y$

Or, $11x-27y=-30$ (3)

From (2), $96-3x+6y=6x+4y$

Or, $2y-9x=-96$ (4)

Multiplying (3) by 9, and (4) by 11,

$$\begin{aligned} 99x-243y &= -\ \ 270 \\ 22y-\ \ 99x &= -1056 \end{aligned}$$

Adding, $-221y=-1326$

$y=6$

Substituting in (4), $12-9x=-96$

$9x=108$

$x=12$

21. $\begin{cases} x-\dfrac{2x+y}{3}=\dfrac{17}{12}-\dfrac{2y+x}{4} & (1) \\ \dfrac{5}{4}-\dfrac{2x-y}{4}=y-\dfrac{2y-x}{3} & (2) \end{cases}$

From (1), $12x-8x-4y=17-6y-3x$

Or, $7x+2y=17$ (3)

From (2), $15-6x+3y=12y-8y+4x$

Or, $10x+y=15$ (4)

Multiplying (4) by 2, $20x+2y=30$

$7x+2y=17$ (3)

Subtracting, $13x=13$

$x=\ 1$

Substituting in (4), $10+y=15$

$y=\ 5$

22. $\begin{cases} \dfrac{2x}{3}-\dfrac{3y}{5}-\dfrac{x+2y}{4}=3-\dfrac{5x-6y}{4} & (1) \\ \dfrac{x}{2}+y-\dfrac{3x-y}{5}=-5+\dfrac{x}{15} & (2) \end{cases}$

From (1), $40x-36y-15x-30y=180-75x+90y$

$100x-156y=180$

Or, $25x-39y=\ 45$ (3)

From (2), $15x+30y-18x+6y=-150+2x$

Or, $36y - 5x = -150$ (4)

Multiplying (4) by 5, $180y - 25x = -750$

$25x - 39y = 45$ (3)

Adding, $141y = -705$

$y = -5$

Substituting in (4), $-180 - 5x = -150$

$-5x = 30$

$x = -6$

23. $\begin{cases} \dfrac{x-2y}{2x-4y-1} = \dfrac{3x}{6x-1} & (1) \\ x - \dfrac{3-5y}{x+2} = \dfrac{4x-13}{4} & (2) \end{cases}$

From (1), $6x^2 - 12xy - x + 2y = 6x^2 - 12xy - 3x$

$2x + 2y = 0$

Or, $y = -x$ (3)

From (2), $4x^2 + 8x - 12 + 20y = 4x^2 - 5x - 26$

Or, $13x + 20y = -14$ (4)

Substituting from (3) in (4), $13x - 20x = -14$

$-7x = -14$

$x = 2$

Substituting in (3), $y = -2$

24. $\begin{cases} 4x^2 + 4xy + 272 = (x+y)(4x+17) & (1) \\ \dfrac{y(x-y)+54}{x-y} = \dfrac{5y+27}{5} & (2) \end{cases}$

From (1), $4x^2 + 4xy + 272 = 4x^2 + 4xy + 17x + 17y$

$17x + 17y = 272$

Or, $x + y = 16$ (3)

From (2), $y + \dfrac{54}{x-y} = y + \dfrac{27}{5}$

$\dfrac{54}{x-y} = \dfrac{27}{5}$

$27(x-y) = 270$

Or, $x - y = 10$ (4)

Adding (3) and (4), $2x = 26$

$x = 13$

Subtracting (4) from (3), $2y = 6$

$y = 3$

25. $\begin{cases} x^2 - 4y^2 - 17 = (x + 2y - 2)(x - 2y + 1) & (1) \\ \dfrac{xy - 5}{y - 2} + \dfrac{1 - 2x}{y - 1} = x & (2) \end{cases}$

From (1), $\quad x^2 - 4y^2 - 17 = x^2 - 4y^2 - x + 6y - 2$

Or, $\quad x - 6y = 15 \quad (3)$

From (2),

$$xy^2 - xy - 5y + 5 + y - 2xy - 2 + 4x = xy^2 - 3xy + 2x$$

Or, $\quad 2x - 4y = -\ 3 \quad (4)$

Multiplying (3) by 2, $\quad 2x - 12y = \ 30 \quad (5)$

Subtracting (5) from (4), $\quad 8y = -33$

$$y = -\frac{33}{8}$$

Substituting in (3), $\quad x + \frac{99}{4} = 15$

$$x = -\frac{39}{4}$$

27. $\begin{cases} 2x - 3y = a & (1) \\ 3x + 4y = b & (2) \end{cases}$

Multiplying (1) by 4, and (2) by 3,

$$8x - 12y = 4a$$
$$9x + 12y = 3b$$

Adding, $\quad 17x = 4a + 3b$

$$x = \frac{4a + 3b}{17}$$

Multiplying (1) by 3, and (2) by 2,

$$6x - 9y = 3a \quad (3)$$
$$6x + 8y = 2b \quad (4)$$

Subtracting (3) from (4),

$$17y = 2b - 3a$$
$$y = \frac{2b - 3a}{17}$$

28. $\begin{cases} ax + by = m & (1) \\ cx + dy = n & (2) \end{cases}$

Multiplying (1) by d, and (2) by b,

$$adx + bdy = dm$$
$$bcx + bdy = bn$$
$$(ad - bc)x = dm - bn$$
$$x = \frac{dm - bn}{ad - bc}$$

Multiplying (1) by c, and (2) by a,

$$acx + bcy = cm \quad (3)$$
$$acx + ady = an \quad (4)$$

Subtracting (3) from (4),

$$(ad - bc)y = an - cm$$
$$y = \frac{an - cm}{ad - bc}$$

29. $\begin{cases} ax - by = c & (1) \\ x - \ y = d & (2) \end{cases}$

Multiplying (2) by b,

$$bx - by = bd \quad (3)$$
$$ax - by = c \quad (1)$$

Subtracting (3) from (1),

$$(a - b)x = c - bd$$
$$x = \frac{c - bd}{a - b}$$

Multiplying (2) by a,

$$ax - ay = ad \quad (4)$$
$$ax - by = c \quad (1)$$

Subtracting (4) from (1),

$$(a - b)y = c - ad$$
$$y = \frac{c - ad}{a - b}$$

30. $\begin{cases} ax - by = 0 & (1) \\ mx + ny = p & (2) \end{cases}$

Multiplying (1) by n, and (2) by b,

$$\begin{aligned} anx - bny &= 0 \\ bmx + bny &= bp \\ \hline (an + bm)x &= bp \end{aligned}$$

$$x = \frac{bp}{an + bm}$$

Multiplying (1) by m, and (2) by a,

$$amx - bmy = 0 \qquad (3)$$

$$amx + any = ap \qquad (4)$$

Subtracting (3) from (4),

$$(an + bm)y = ap$$

$$y = \frac{ap}{an + bm}$$

31. $\begin{cases} ax + by = m & (1) \\ cx - dy = n & (2) \end{cases}$

Multiplying (1) by d, and (2) by b,

$$\begin{aligned} adx + bdy &= dm \\ bcx - bdy &= bn \\ \hline \end{aligned}$$

Adding, $(ad + bc)x = dm + bn$

$$x = \frac{dm + bn}{ad + bc}$$

Multiplying (1) by c, and (2) by a,

$$\begin{aligned} acx + bcy &= cm \\ acx - ady &= an \\ \hline (ad + bc)y &= cm - an \end{aligned}$$

$$y = \frac{cm - an}{ad + bc}$$

32. $\begin{cases} mx - ny = p & (1) \\ m'x - n'y = p' & (2) \end{cases}$

Multiplying (1) by n', and (2) by n,

$$\begin{aligned} mn'x - nn'y &= n'p \\ m'nx - nn'y &= np' \\ \hline \end{aligned}$$

Subtracting, $(mn' - m'n)x = n'p - np'$

$$x = \frac{n'p - np'}{mn' - m'n}$$

Multiplying (1) by m', and (2) by m,

$$\begin{aligned} mm'x - m'ny &= m'p \\ mm'x - mn'y &= mp' \\ \hline \end{aligned}$$

Subtracting, $(mn' - m'n)y = m'p - mp'$

$$y = \frac{m'p - mp'}{mn' - m'n}$$

33. $\begin{cases} \dfrac{x}{a} - \dfrac{y}{b} = m & (1) \\[2ex] \dfrac{x}{c} + \dfrac{y}{d} = n & (2) \end{cases}$

From (1), $bx - ay = abm \qquad (3)$

From (2), $dx + cy = cdn \qquad (4)$

Multiplying (3) by c, and (4) by a,

$$\begin{aligned} bcx - acy &= abcm \\ adx + acy &= acdn \\ \hline \end{aligned}$$

Adding, $(ad + bc)x = abcm + acdn$

$$x = \frac{ac(bm + dn)}{ad + bc}$$

Multiplying (3) by d, and (4) by b,

$$bdx - ady = abdm \quad (5)$$
$$bdx + bcy = bcdn \quad (6)$$

Subtracting (5) from (6),

$$(ad + bc)y = bcdn - abdm$$
$$y = \frac{bd(cn - am)}{ad + bc}$$

34. $\begin{cases} x + ay = a(a + 2b) & (1) \\ y - \dfrac{x}{b} = b & (2) \end{cases}$

From (1), $\quad x + ay = a^2 + 2ab \quad (3)$

From (2), $\quad by - x = b^2$

Adding, $\quad ay + by = a^2 + 2ab + b^2$

$$(a + b)y = (a + b)^2$$
$$y = a + b$$

Substituting in (3), $\quad x + a^2 + ab = a^2 + 2ab$

$$x = ab$$

35. $\begin{cases} ax + by = 2 & (1) \\ ab(ay - bx) = a^2 - b^2 & (2) \end{cases}$

From (2), $\quad a^2by - ab^2x = a^2 - b^2 \quad (3)$

Multiplying (1) by a^2, $\quad a^3x + a^2by = 2a^2$

$$a^2by - ab^2x = a^2 - b^2 \quad (3)$$

Subtracting, $\quad a^3x + ab^2x = a^2 + b^2$

$$ax(a^2 + b^2) = a^2 + b^2$$
$$x = \frac{1}{a}$$

Substituting in (1), $\quad 1 + by = 2$

$$by = 1$$
$$y = \frac{1}{b}$$

36. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 2ab & (1) \\ x + y = ab(a + b) & (2) \end{cases}$

From (1), $\quad bx + ay = 2a^2b^2 \quad (3)$

From (2), $\quad x + y = a^2b + ab^2 \quad (4)$

Multiplying (4) by a, $\quad ax + ay = a^3b + a^2b^2$

$$bx + ay = 2a^2b^2 \qquad (3)$$

Subtracting, $\quad (a-b)x = a^3b - a^2b^2$

$$= a^2b(a-b)$$

$$x = a^2b$$

Substituting in (4), $\quad a^2b + y = a^2b + ab^2$

$$y = ab^2$$

37. $\begin{cases} mx + ny = \dfrac{m^4+n^4}{m^2n^2} & (1) \\ nx + my = \dfrac{m^2+n^2}{mn} & (2) \end{cases}$

From (1), $\quad m^3n^2x + m^2n^3y = m^4 + n^4 \qquad (3)$

From (2), $\quad mn^2x + m^2ny = m^2 + n^2 \qquad (4)$

Multiplying (4) by n^2,

$$mn^4x + m^2n^3y = m^2n^2 + n^4 \qquad (5)$$

Subtracting (5) from (3),

$$m^3n^2x - mn^4x = m^4 - m^2n^2$$

$$mn^2x(m^2-n^2) = m^2(m^2-n^2)$$

$$x = \frac{m}{n^2}$$

Substituting in (4),

$$m^2 + m^2ny = m^2 + n^2$$

$$m^2ny = n^2$$

$$y = \frac{n}{m^2}$$

38. $\begin{cases} (a+b)x - (a-b)y = 4ab & (1) \\ (a-b)x - (a+b)y = 0 & (2) \end{cases}$

Multiplying (1) by $a+b$, and (2) by $a-b$,

$$(a+b)^2x - (a^2-b^2)y = 4ab(a+b)$$

$$(a-b)^2x - (a^2-b^2)y = 0$$

Subtracting, $\quad \{(a+b)^2 - (a-b)^2\}x = 4ab(a+b)$

$$4abx = 4ab(a+b)$$

$$x = a+b$$

Substituting in (2),

$$a^2 - b^2 - (a+b)y = 0$$

$$(a+b)y = a^2 - b^2$$

$$y = \frac{a^2-b^2}{a+b} = a-b$$

39. $\begin{cases} \dfrac{x+a}{b}+\dfrac{y+b}{a}=\dfrac{2(a^2+b^2)}{ab} & (1) \\ \dfrac{x-b}{a}-\dfrac{y-a}{b}=\dfrac{a^2-b^2}{ab} & (2) \end{cases}$

From (1), $ax+a^2+by+b^2=2a^2+2b^2$

Or, $ax+by=a^2+b^2$ (3)

From (2), $bx-b^2-ay+a^2=a^2-b^2$

Or, $bx-ay=0$ (4)

Multiplying (3) by a, and (4) by b,

$$a^2x+aby=a(a^2+b^2)$$
$$b^2x-aby=0$$

Adding, $(a^2+b^2)x=a(a^2+b^2)$

$$x=a$$

Substituting in (4), $y=\dfrac{ba}{a}=b$

40. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{m+n}=\dfrac{a^2+m^2-n^2}{a(m+n)} & (1) \\ (m+n)^2(m-n)x=a^3y & (2) \end{cases}$

From (1), $(m+n)x+ay=a^2+m^2-n^2$ (3)

Multiplying by a^2, $a^2(m+n)x+a^3y=a^2(a^2+m^2-n^2)$

By (2), $(m+n)^2(m-n)x-a^3y=0$

Adding, $(m+n)x\{a^2+(m+n)(m-n)\}=a^2(a^2+m^2-n^2)$

$$(m+n)x(a^2+m^2-n^2)=a^2(a^2+m^2-n^2)$$

$$x=\frac{a^2}{m+n}$$

Substituting in (2), $(m+n)(m-n)a^2=a^3y$

$$y=\frac{m^2-n^2}{a}$$

41. $\begin{cases} \dfrac{x}{a+b}+\dfrac{y}{a-b}=\dfrac{1}{a^2-b^2} & (1) \\ \dfrac{x}{a-b}+\dfrac{y}{a+b}=\dfrac{1}{a^2-b^2} & (2) \end{cases}$

From (1), $(a-b)x+(a+b)y=1$ (3)

From (2), $(a+b)x+(a-b)y=1$ (4)

Adding (3) and (4), $2ax+2ay=2$

$$x+y=\frac{1}{a} \quad (5)$$

Subtracting (3) from (4), $2bx - 2by = 0$

$$x - y = 0 \qquad (6)$$

Adding (5) and (6), $2x = \frac{1}{a}$

$$x = \frac{1}{2a}$$

Substituting in (6), $y = x = \frac{1}{2a}$

42. $\begin{cases} \dfrac{x}{a+b} + \dfrac{y}{a-b} = 2a & (1) \\ x - y = 4ab & (2) \end{cases}$

From (1), $(a-b)x + (a+b)y = 2a^3 - 2ab^2 \qquad (3)$

Multiplying (2) by $a+b$,

$$(a+b)x - (a+b)y = 4a^2b + 4ab^2 \qquad (4)$$

Adding (3) and (4), $x[a+b+a-b] = 2a^3 + 4a^2b + 2ab^2$

$$2ax = 2a(a+b)^2$$

$$x = (a+b)^2$$

Substituting in (1), $a + b + \dfrac{y}{a-b} = 2a$

$$\frac{y}{a-b} = a - b$$

$$y = (a-b)^2$$

44. $\begin{cases} \dfrac{8}{x} + \dfrac{1}{y} = \dfrac{5}{4} & (1) \\ \dfrac{2}{x} - \dfrac{3}{y} = -1 & (2) \end{cases}$

Multiplying (1) by 4, and (2) by 6,

$$\frac{12}{x} + \frac{4}{y} = 5 \qquad (3)$$

$$\frac{12}{x} - \frac{18}{y} = -6 \qquad (4)$$

Subtracting (4) from (3),

$$\frac{22}{y} = 11$$

$$11y = 22$$

$$y = 2$$

Substituting in (3),

$$\frac{12}{x} + 2 = 5$$

$$\frac{12}{x} = 3$$

$$x = 4$$

45. $\begin{cases} \dfrac{2}{x} - \dfrac{3}{y} = -\dfrac{7}{5} & (1) \\ \dfrac{15}{x} - \dfrac{8}{y} = -\dfrac{17}{3} & (2) \end{cases}$

Multiplying (1) by 45, and (2) by 6,

$$\frac{90}{x} - \frac{135}{y} = -63 \qquad (3)$$

$$\frac{90}{x} - \frac{48}{y} = -34 \qquad (4)$$

Subtracting (3) from (4),

$$\frac{87}{y} = 29$$

$$y = 3$$

Substituting in (1),

$$\frac{2}{x} - 1 = -\frac{7}{5}$$

$$\frac{2}{x} = -\frac{2}{5}$$

$$x = -5$$

46. $$\begin{cases} \dfrac{11}{x}-\dfrac{7}{y}=\dfrac{3}{2} & (1) \\ \dfrac{2}{x}+\dfrac{4}{y}=-5 & (2) \end{cases}$$

Multiplying (1) by 2, and (2) by 11,

$$\frac{22}{x}-\frac{14}{y}=3 \qquad (3)$$

$$\frac{22}{x}+\frac{44}{y}=-55 \qquad (4)$$

Subtracting (3) from (4),

$$\frac{58}{y}=-58$$

$$y=-1$$

Substituting in (2),

$$\frac{2}{x}-4=-5$$

$$\frac{2}{x}=-1$$

$$x=-2$$

47. $$\begin{cases} \dfrac{3}{x}-\dfrac{5}{2y}=16 & (1) \\ \dfrac{1}{2x}+\dfrac{4}{y}=-15 & (2) \end{cases}$$

Multiplying (1) by 2, and (2) by 12,

$$\frac{6}{x}-\frac{5}{y}=32 \qquad (3)$$

$$\frac{6}{x}+\frac{48}{y}=-180 \qquad (4)$$

Subtracting (3) from (4),

$$\frac{53}{y}=-212$$

$$y=-\frac{53}{212}=-\frac{1}{4}$$

Substituting in (1),

$$\frac{3}{x}+10=16$$

$$\frac{3}{x}=6$$

$$x=\frac{3}{6}=\frac{1}{2}$$

48. $$\begin{cases} \dfrac{m}{x}+\dfrac{n}{y}=1 & (1) \\ \dfrac{n}{x}+\dfrac{m}{y}=1 & (2) \end{cases}$$

Multiplying (1) by m, and (2) by n,

$$\frac{m^2}{x}+\frac{mn}{y}=m$$

$$\frac{n^2}{x}+\frac{mn}{y}=n$$

Subtracting,

$$\frac{m^2-n^2}{x}=m-n$$

$$x=\frac{m^2-n^2}{m-n}=m+n$$

Multiplying (1) by n, and (2) by m,

$$\frac{mn}{x}+\frac{n^2}{y}=n \qquad (3)$$

$$\frac{mn}{x}+\frac{m^2}{y}=m \qquad (4)$$

Subtracting (3) from (4),

$$\frac{m^2-n^2}{y}=m-n$$

$$y=\frac{m^2-n^2}{m-n}=m+n$$

49. $$\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=m & (1) \\ \dfrac{c}{x}+\dfrac{d}{y}=n & (2) \end{cases}$$

Multiplying (1) by d, and (2) by b,

$$\frac{ad}{x}+\frac{bd}{y}=dm \qquad (3)$$

$$\frac{bc}{x}+\frac{bd}{y}=bn \qquad (4)$$

Subtracting (3) from (4)

$$\frac{bc-ad}{x}=bn-dm$$

$$x=\frac{bc-ad}{bn-dm}$$

Multiplying (1) by c, and (2) by a,

$$\frac{ac}{x}+\frac{bc}{y}=cm$$

$$\frac{ac}{x}+\frac{ad}{y}=an$$

Subtracting,

$$\frac{bc-ad}{y}=cm-an$$

$$y=\frac{bc-ad}{cm-an}$$

50. $\begin{cases} \dfrac{2}{9x}-\dfrac{5}{2y}=-3 & (1) \\ \dfrac{5}{3x}+\dfrac{1}{4y}=\dfrac{17}{6} & (2) \end{cases}$

Multiplying (1) by 90, and (2) by 12,

$$\frac{20}{x}-\frac{225}{y}=-270 \qquad (3)$$

$$\frac{20}{x}+\frac{3}{y}=34 \qquad (4)$$

Subtracting (3) from (4),

$$\frac{228}{y}=304$$

$$y=\frac{228}{304}=\frac{3}{4}$$

Substituting in (4),

$$\frac{20}{x}+4=34$$

$$\frac{20}{x}=30$$

$$x=\frac{20}{30}=\frac{2}{3}$$

51. $\begin{cases} \dfrac{m^2}{x}+\dfrac{n^2}{y}=m^2n+mn^2 & (1) \\ \dfrac{n}{x}+\dfrac{m}{y}=m^2+n^2 & (2) \end{cases}$

Multiplying (1) by m, and (2) by n^2,

$$\frac{m^3}{x}+\frac{mn^2}{y}=m^3n+m^2n^2 \qquad (3)$$

$$\frac{n^3}{x}+\frac{mn^2}{y}=m^2n^2+n^4 \qquad (4)$$

Subtracting (4) from (3),

$$\frac{m^3-n^3}{x}=m^3n-n^4$$

$$=n(m^3-n^3)$$

$$x=\frac{m^3-n^3}{n(m^3-n^3)}=\frac{1}{n}$$

Substituting in (2),

$$n^2+\frac{m}{y}=m^2+n^2$$

$$y=\frac{m}{m^2}=\frac{1}{m}$$

CHAPTER XV.

Art. 189. — Pages 145–147.

3.
$$\begin{cases} x+y=2 & (1)\\ y+z=-1 & (2)\\ z+x=3 & (3)\end{cases}$$

Adding,
$$2x+2y+2z=4$$
Or,
$$x+y+z=2 \qquad (4)$$
Subtracting (2) from (4),
$$x=3$$
Subtracting (3) from (4),
$$y=-1$$
Subtracting (1) from (4),
$$z=0$$

4.
$$\begin{cases} 2x-5y=-19 & (1)\\ 3y+4z=13 & (2)\\ 2z-5x=12 & (3)\end{cases}$$

Multiplying (3) by 2,
$$\begin{aligned} 4z-10x&=24\\ 3y+4z&=13 \qquad (2)\end{aligned}$$
Subtracting,
$$-10x-3y=11 \qquad (4)$$
Multiplying (1) by 5,
$$\begin{aligned} 10x-25y&=-95\\ -10x-3y&=11 \qquad (4)\end{aligned}$$
Adding,
$$\begin{aligned} -28y&=-84\\ y&=3\end{aligned}$$
Substituting in (1),
$$\begin{aligned} 2x-15&=-19\\ 2x&=-4\\ x&=-2\end{aligned}$$
Substituting in (3),
$$\begin{aligned} 2z+10&=12\\ z&=1\end{aligned}$$

5.
$$\begin{cases} 3x-2y=-1 & (1)\\ 5y+4z=-6 & (2)\\ x-y-3z=11 & (3)\end{cases}$$

Multiplying (3) by 3,
$$\begin{aligned} 3x-3y-9z&=33\\ 3x-2y&=-1 \qquad (1)\end{aligned}$$
Subtracting,
$$-y-9z=34 \qquad (4)$$
Multiplying (4) by 5,
$$\begin{aligned} -5y-45z&=170\\ 5y+4z&=-6 \qquad (2)\end{aligned}$$
Adding,
$$\begin{aligned} -41z&=164\\ z&=-4\end{aligned}$$
Substituting in (2),
$$\begin{aligned} 5y-16&=-6\\ 5y&=10\\ y&=2\end{aligned}$$
Substituting in (1),
$$\begin{aligned} 3x-4&=-1\\ 3x&=3\\ x&=1\end{aligned}$$

6.
$$\begin{cases} 2x-y=5 & (1)\\ 3x+2y-z=6 & (2)\\ x-3y+2z=1 & (3)\end{cases}$$

Multiplying (2) by 2,
$$\begin{aligned} 6x+4y-2z&=12\\ 2x-y&=5 \qquad (1)\\ x-3y+2z&=1 \qquad (3)\end{aligned}$$
Adding,
$$\begin{aligned} 9x&=18\\ x&=2\end{aligned}$$

Substituting in (1),

$$4 - y = 5$$
$$y = -1$$

Substituting in (2),

$$6 - 2 - z = 6$$
$$z = -2$$

7. $\begin{cases} x + y + z = 53 & (1) \\ x + 2y + 3z = 107 & (2) \\ x + 3y + 4z = 137 & (3) \end{cases}$

Subtracting (1) from (2),

$$y + 2z = 54 \qquad (4)$$

Subtracting (2) from (3),

$$y + z = 30 \qquad (5)$$

Subtracting (5) from (4),

$$z = 24$$

Substituting in (5),

$$y + 24 = 30$$
$$y = 6$$

Substituting in (1),

$$x + 6 + 24 = 53$$
$$x = 23$$

8. $\begin{cases} 3x - y - 2z = -23 & (1) \\ 6x + 2y + 3z = 15 & (2) \\ 4x + 3y - z = -6 & (3) \end{cases}$

Adding, $13x + 4y = -14 \qquad (4)$

Multiplying (3) by 2,

$$8x + 6y - 2z = -12$$
$$3x - y - 2z = -23 \qquad (1)$$

Subtracting,

$$5x + 7y = 11 \qquad (5)$$

Multiplying (4) by 7, and (5) by 4,

$$91x + 28y = -98$$
$$20x + 28y = 44$$

Subtracting,

$$71x = -142$$
$$x = -2$$

Substituting in (5),

$$-10 + 7y = 11$$
$$7y = 21$$
$$y = 3$$

Substituting in (3),

$$-8 + 9 - z = -6$$
$$z = 7$$

9. $\begin{cases} x + y - z = 3 & (1) \\ y + z - x = 1 & (2) \\ z + x - y = -11 & (3) \end{cases}$

Adding (1) and (3),

$$2x = -8$$
$$x = -4$$

Adding (1) and (2),

$$2y = 4$$
$$y = 2$$

Adding (2) and (3),

$$2z = -10$$
$$z = -5$$

10. $\begin{cases} x - 2y + 3z = 0 & (1) \\ y - 2z + 3x = -25 & (2) \\ z - 2x + 3y = 9 & (3) \end{cases}$

Multiplying (3) by 2,

$$2z - 4x + 6y = 18$$
$$y - 2z + 3x = -25 \qquad (2)$$

Adding, $7y - x = -7 \qquad (4)$

Multiplying (3) by 3,

$$3z - 6x + 9y = 27$$
$$x - 2y + 3z = 0 \qquad (1)$$

Subtracting,

$$11y - 7x = 27 \qquad (5)$$

Multiplying (4) by 7,

$$49y - 7x = -49$$
$$11y - 7x = 27 \qquad (5)$$

Subtracting, $38y = -76$

$$y = -2$$

Substituting in (4),

$$-14 - x = -7$$
$$x = -7$$

Substituting in (3),

$$z + 14 - 6 = 9$$
$$z = 1$$

11. $\begin{cases} 5x-3y+2z= & 41 & (1) \\ 2x+y-z= & 17 & (2) \\ 5x+4y-2z= & 36 & (3) \end{cases}$

Adding (1) and (3),

$$10x+y=77 \quad (4)$$

Multiplying (2) by 2,

$$4x+2y-2z=34$$
$$5x-3y+2z=41 \quad (1)$$

Adding, $9x-y=75 \quad (5)$

Adding (4) and (5),

$$19x=152$$
$$x=8$$

Substituting in (4),

$$80+y=77$$
$$y=-3$$

Substituting in (2),

$$16-3-z=17$$
$$z=-4$$

12. $\begin{cases} 2x+y+z=-2 & (1) \\ x+2y+z=0 & (2) \\ x+y+2z=-4 & (3) \end{cases}$

Adding,

$$4x+4y+4z=-6$$

Or, $x+y+z=-\frac{3}{2} \quad (4)$

Subtracting (4) from (1),

$$x=-\frac{1}{2}$$

Subtracting (4) from (2),

$$y=\frac{3}{2}$$

Subtracting (4) from (3),

$$z=-\frac{5}{2}$$

13. $\begin{cases} 7x+4y-z=-50 & (1) \\ 4x-5y-3z=20 & (2) \\ x-3y-4z=30 & (3) \end{cases}$

Multiplying (1) by 3,

$$21x+12y-3z=-150$$
$$4x-5y-3z=20 \quad (2)$$

Subtracting,

$$17x+17y=-170$$

Or, $x+y=-10 \quad (4)$

Multiplying (1) by 4,

$$28x+16y-4z=-200$$
$$x-3y-4z=30 \quad (3)$$

Subtracting,

$$27x+19y=-230 \quad (5)$$

Multiplying (4) by 19,

$$19x+19y=-190 \quad (6)$$

Subtracting (6) from (5),

$$8x=-40$$
$$x=-5$$

Substituting in (4),

$$-5+y=-10$$
$$y=-5$$

Substituting in (1),

$$-35-20-z=-50$$
$$z=-5$$

14. $\begin{cases} x-6y+4z=3 & (1) \\ 4x+4y-3z=10 & (2) \\ 2x+y+6z=46 & (3) \end{cases}$

Multiplying (1) by 2,

$$2x-12y+8z=6$$
$$2x+y+6z=46 \quad (3)$$

Subtracting,

$$-13y+2z=-40 \quad (4)$$

Multiplying (1) by 4,

$$4x-24y+16z=12$$
$$4x+4y-3z=10 \quad (2)$$

Subtracting,

$$-28y+19z=2 \quad (5)$$

Multiplying (4) by 19, and (5) by 2,

$$-247y+38z=-760 \quad (6)$$
$$-56y+38z=4 \quad (7)$$

Subtracting (6) from (7),

$$191y=764$$
$$y=4$$

Substituting in (4),

$$-52+2z=-40$$
$$2z=12$$
$$z=6$$

Substituting in (1),

$$x-24+24=3$$
$$x=3$$

15. $\begin{cases} 8x-9y-7z=-36 & (1) \\ 12x-y-3z=36 & (2) \\ 6x-2y-z=10 & (3) \end{cases}$

Multiplying (3) by 7,

$$42x-14y-7z=70$$
$$8x-9y-7z=-36 \quad (1)$$

Subtracting,

$$34x-5y=106 \quad (4)$$

Multiplying (3) by 3,

$$18x-6y-3z=30$$
$$12x-y-3z=36 \quad (2)$$

Subtracting,

$$6x-5y=-6 \quad (5)$$

Subtracting (5) from (4),

$$28x=112$$
$$x=4$$

Substituting in (5),

$$24-5y=-6$$
$$5y=30$$
$$y=6$$

Substituting in (3),

$$24-12-z=10$$
$$z=2$$

16. $\begin{cases} 4x-3y+2z=40 & (1) \\ 5x+9y-7z=47 & (2) \\ 9x+8y-3z=97 & (3) \end{cases}$

Multiplying (1) by 3,

$$12x-9y+6z=120$$
$$5x+9y-7z=47 \quad (2)$$

Adding,

$$17x-z=167 \quad (4)$$

Multiplying (1) by 8, and (3) by 3,

$$32x-24y+16z=320$$
$$27x+24y-9z=291$$

Adding,

$$59x+7z=611 \quad (5)$$

Multiplying (4) by 7,

$$119x-7z=1169 \quad (6)$$

Adding (5) and (6),

$$178x=1780$$
$$x=10$$

Substituting in (4),

$$170-z=167$$
$$z=3$$

Substituting in (1),

$$40-3y+6=40$$
$$3y=6$$
$$y=2$$

17. $\begin{cases} \frac{x}{2}+\frac{y}{3}-\frac{z}{4}=-43 & (1) \\ \frac{x}{3}-\frac{y}{4}+\frac{z}{2}=34 & (2) \\ \frac{x}{4}+\frac{y}{2}-\frac{z}{3}=-50 & (3) \end{cases}$

Multiplying (1) by 24, and (2) by 12,

$$12x+8y-6z=-1032$$
$$4x-3y+6z=408$$

Adding,

$$16x+5y=-624 \quad (4)$$

Multiplying (2) by 24, and (3) by 36,

$$8x-6y+12z=816$$
$$9x+18y-12z=-1800$$

Adding,

$$17x+12y=-984 \quad (5)$$

Multiplying (4) by 12, and (5) by 5,

$$192x+60y=-7488$$
$$85x+60y=-4920$$

Subtracting,

$$107x=-2568$$
$$x=-24$$

Substituting in (4),

$$-384+5y=-624$$
$$5y=-240$$
$$y=-48$$

Substituting in (2),

$$-8+12+\frac{z}{2}=34$$

$$\frac{z}{2}=30$$

$$z=60$$

18. $\begin{cases} 2u-3x=1 & (1) \\ 3x-4y=-1 & (2) \\ 4y-5z=1 & (3) \\ 5z-6u=-2 & (4) \end{cases}$

Adding, $-4u=-1$

$$u=\frac{1}{4}$$

Substituting in (1),

$$\frac{1}{2}-3x=1$$

$$3x=-\frac{1}{2}$$

$$x=-\frac{1}{6}$$

Substituting in (2),

$$-\frac{1}{2}-4y=-1$$

$$4y=\frac{1}{2}$$

$$y=\frac{1}{8}$$

Substituting in (3),

$$\frac{1}{2}-5z=1$$

$$5z=-\frac{1}{2}$$

$$z=-\frac{1}{10}$$

19. $\begin{cases} 2y+z+2u=-23 & (1) \\ y+3z=-2 & (2) \\ 4x+z=13 & (3) \\ \frac{x}{3}+3u=-20 & (4) \end{cases}$

Multiplying (2) by 2,

$$2y+6z=-4$$

$$2y+z+2u=-23 \quad (1)$$

Subtracting,

$$5z-2u=19 \quad (5)$$

Multiplying (4) by 12,

$$4x+36u=-240$$

$$4x+z=13 \quad (3)$$

Subtracting,

$$36u-z=-253 \quad (6)$$

Multiplying (5) by 18,

$$90z-36u=342 \quad (7)$$

Adding (6) and (7),

$$89z=89$$

$$z=1$$

Substituting in (5),

$$5-2u=19$$

$$2u=-14$$

$$u=-7$$

Substituting in (3),

$$4x+1=13$$

$$4x=12$$

$$x=3$$

Substituting in (2),

$$y+3=-2$$

$$y=-5$$

20. $\begin{cases} \frac{1}{x}+\frac{1}{y}=1 & (1) \\ \frac{1}{y}+\frac{1}{z}=\frac{3}{2} & (2) \\ \frac{1}{z}+\frac{1}{x}=2 & (3) \end{cases}$

Adding,

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{9}{2}$$

Or, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{9}{4} \quad (4)$

Subtracting (2) from (4),

$$\frac{1}{x}=\frac{3}{4}$$

$$x=\frac{4}{3}$$

Subtracting (3) from (4),

$$\frac{1}{y}=\frac{1}{4}$$

$$y=4$$

Subtracting (1) from (4),

$$\frac{1}{z}=\frac{5}{4}$$

$$z=\frac{4}{5}$$

21. $$\begin{cases} \frac{3}{x}-\frac{2}{y}=-13 & (1) \\ \frac{3}{y}+\frac{2}{z}=\quad 14 & (2) \\ \frac{3}{z}-\frac{2}{x}=\quad 18 & (3) \end{cases}$$

Multiplying (2) by 3, and (3) by 2,

$$\frac{9}{y}+\frac{6}{z}=\quad 42$$

$$\frac{6}{z}-\frac{4}{x}=\quad 36$$

Subtracting,

$$\frac{9}{y}+\frac{4}{x}=\quad 6 \qquad (4)$$

Multiplying (1) by 4, and (4) by 3,

$$\frac{12}{x}-\frac{8}{y}=-52 \qquad (5)$$

$$\frac{27}{y}+\frac{12}{x}=\quad 18 \qquad (6)$$

Subtracting (5) from (6),

$$\frac{35}{y}=\quad 70$$

$$y=\frac{35}{70}=\frac{1}{2}$$

Substituting in (1),

$$\frac{3}{x}-4=-13$$

$$\frac{3}{x}=-\ 9$$

$$x=-\frac{3}{9}=-\frac{1}{3}$$

Substituting in (2),

$$6+\frac{2}{z}=14$$

$$\frac{2}{z}=\ 8$$

$$z=\ \frac{2}{8}=\frac{1}{4}$$

22. $$\begin{cases} ax\ +a^2y=2 & (1) \\ a^2y+a^3z=2 & (2) \\ a^3z+a^4x=a^3+1 & (3) \end{cases}$$

Subtracting (2) from (3),

$$a^4x-a^2y=a^3-1 \qquad (4)$$

Adding (1) and (4),

$$a^4x+ax=a^3+1$$

$$ax(a^3+1)=a^3+1$$

$$x=\frac{1}{a}$$

Substituting in (1),

$$1+a^2y=2$$

$$a^2y=1$$

$$y=\frac{1}{a^2}$$

Substituting in (2),

$$1+a^3z=2$$

$$a^3z=1$$

$$z=\frac{1}{a^3}$$

23. $$\begin{cases} 3u+x+2y-z=\ 22 & (1) \\ 4x-y+3z=\ 35 & (2) \\ 4u+3x-2y=\ 19 & (3) \\ 2u+4y+z=\ 39 & (4) \end{cases}$$

Adding (1) and (4),

$$5u+x+6y=\ 61 \qquad (5)$$

Multiplying (1) by 3,

$$9u+3x+6y-3z=\ 66$$

$$4x-\ \ y+3z=\ 35 \qquad (2)$$

Adding,

$$9u+7x+5y=101 \qquad (6)$$

Multiplying (5) by 7,

$$35u+7x+42y=427 \qquad (7)$$

Subtracting (6) from (7),

$$26u + 37y = 326 \quad (8)$$

Multiplying (5) by 3,

$$15u + 3x + 18y = 183$$
$$4u + 3x - 2y = 19 \quad (3)$$

Subtracting,

$$11u + 20y = 164 \quad (9)$$

Multiplying (8) by 11, and (9) by 26,

$$286u + 407y = 3586 \quad (10)$$
$$286u + 520y = 4264 \quad (11)$$

Subtracting (10) from (11),

$$113y = 678$$
$$y = 6$$

Substituting in (9),

$$11u + 120 = 164$$
$$11u = 44$$
$$u = 4$$

Substituting in (5),

$$20 + x + 36 = 61$$
$$x = 5$$

Substituting in (4),

$$8 + 24 + z = 39$$
$$z = 7$$

24. $$\begin{cases} \frac{1}{x} + \frac{2}{y} + \frac{3}{z} = -7 & (1) \\ \frac{2}{x} - \frac{3}{y} + \frac{4}{z} = 9 & (2) \\ \frac{3}{x} + \frac{1}{y} - \frac{2}{z} = 5 & (3) \end{cases}$$

Adding,

$$\frac{6}{x} + \frac{5}{z} = 7 \quad (4)$$

Multiplying (3) by 2,

$$\frac{6}{x} + \frac{2}{y} - \frac{4}{z} = 10$$
$$\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = -7 \quad (1)$$

Subtracting,

$$\frac{5}{x} - \frac{7}{z} = 17 \quad (5)$$

Multiplying (4) by 5, and (5) by 6,

$$\frac{30}{x} + \frac{25}{z} = 35$$
$$\frac{30}{x} - \frac{42}{z} = 102$$

Subtracting,

$$\frac{67}{z} = -67$$
$$z = -1$$

Substituting in (4),

$$\frac{6}{x} - 5 = 7$$
$$\frac{6}{x} = 12$$
$$x = \frac{6}{12} = \frac{1}{2}$$

Substituting in (3),

$$6 + \frac{1}{y} + 2 = 5$$
$$\frac{1}{y} = -3$$
$$y = -\frac{1}{3}$$

25. $$\begin{cases} y - z - \frac{x+z}{2} = 1 & (1) \\ \frac{x-y}{5} - \frac{x-z}{6} = 0 & (2) \\ \frac{y+z}{4} - \frac{x+y}{2} = -4 & (3) \end{cases}$$

From (1), $$2y - 2z - x - z = 2$$

Or, $$-x + 2y - 3z = 2 \quad (4)$$

From (2), $6x - 6y - 5x + 5z = 0$

Or, $x - 6y + 5z = 0$ (5)

From (3), $y + z - 2x - 2y = -16$

Or, $-2x - y + z = -16$ (6)

Adding (4) and (5), $-4y + 2z = 2$

Or, $-2y + z = 1$ (7)

Multiplying (4) by 2, $-2x + 4y - 6z = 4$

$-2x - y + z = -16$ (6)

Subtracting, $5y - 7z = 20$ (8)

Multiplying (7) by 7, $-14y + 7z = 7$ (9)

Adding (8) and (9), $-9y = 27$

$y = -3$

Substituting in (7), $6 + z = 1$

$z = -5$

Substituting in (5), $x + 18 - 25 = 0$

$x = 7$

26. $\begin{cases} ay + bx = c & (1) \\ cx + az = b & (2) \\ bz + cy = a & (3) \end{cases}$

Multiplying (1) by $2c$, (2) by $2b$, and (3) by $2a$,

$2cay + 2bcx = 2c^2$ (4)

$2bcx + 2abz = 2b^2$ (5)

$2abz + 2cay = 2a^2$ (6)

Adding, $4bcx + 4cay + 4abz = 2a^2 + 2b^2 + 2c^2$

Or, $2bcx + 2cay + 2abz = a^2 + b^2 + c^2$ (7)

Subtracting (6), (5), and (4) from (7),

$2bcx = b^2 + c^2 - a^2$

$2cay = c^2 + a^2 - b^2$

$2abz = a^2 + b^2 - c^2$

Whence, $x = \dfrac{b^2 + c^2 - a^2}{2bc}$

$y = \dfrac{c^2 + a^2 - b^2}{2ca}$

$z = \dfrac{a^2 + b^2 - c^2}{2ab}$

27. $\begin{cases} 2 - z - \dfrac{3x + y}{4} = 0 & (1) \\ 8 - \dfrac{y + 16z}{3} = 3x & (2) \\ 25 - 12(x + z) = -y & (3) \end{cases}$

From (1), $8-4z-3x-y=0$

Or, $3x+y+4z=8 \qquad (4)$

From (2), $24-y-16z=9x$

Or, $9x+y+16z=24 \qquad (5)$

From (3), $25-12x-12z=-y$

Or, $12x-y+12z=25 \qquad (6)$

Subtracting (4) from (5), $6x+12z=16$

Or, $3x+6z=8 \qquad (7)$

Adding (5) and (6), $21x+28z=49$

Or, $3x+4z=7 \qquad (8)$

Subtracting (8) from (7), $2z=1$

$z=\frac{1}{2}$

Substituting in (7), $3x+3=8$

$x=1\frac{2}{3}$

Substituting in (4), $5+y+2=8$

$y=1$

28. $$\begin{cases} \dfrac{2x+y}{4}-\dfrac{y-2z}{3}=1 & (1) \\[2ex] \dfrac{x+3y}{8}-\dfrac{x-z}{4}=-2 & (2) \\[2ex] \dfrac{z+y}{3}-\dfrac{z+x}{4}=-\dfrac{3}{2} & (3) \end{cases}$$

From (1), $6x+3y-4y+8z=12$

Or, $6x-y+8z=12 \qquad (4)$

From (2), $4x+12y-3x+3z=-24$

Or, $x+12y+3z=-24 \qquad (5)$

From (3), $4z+4y-3z-3x=-18$

Or, $-3x+4y+z=-18 \qquad (6)$

Multiplying (6) by 2, $-6x+8y+2z=-36$

$6x-y+8z=12 \qquad (4)$

Adding, $7y+10z=-24 \qquad (7)$

Multiplying (5) by 3, $3x+36y+9z=-72$

$-3x+4y+z=-18 \qquad (6)$

Adding, $40y+10z=-90 \qquad (8)$

Subtracting (7) from (8), $33y=-66$

$y=-2$

Substituting in (7), $-14+10z=-24$

$z=-1$

Substituting in (5), $x-24-3=-24$

$x=3$

29. $\begin{cases} ax + y - z = a^2 + a - 1 & (1) \\ ay + z - x = a^2 - a + 1 & (2) \\ az + x - y = a & (3) \end{cases}$

Adding, $\quad ax + ay + az = 2a^2 + a \quad (4)$

Or, $\quad x + y + z = 2a + 1 \quad (5)$

Adding (1) and (5), $\quad (a + 1)x + 2y = a^2 + 3a \quad (6)$

Subtracting (3) from (4),

$$(a - 1)x + (a + 1)y = 2a^2 \quad (7)$$

Multiplying (6) by $a + 1$, and (7) by 2,

$$(a^2 + 2a + 1)x + 2(a + 1)y = a^3 + 4a^2 + 3a$$
$$(2a - 2)x + 2(a + 1)y = 4a^2$$

Subtracting, $\quad (a^2 + 3)x = a^3 + 3a$

$$= a(a^2 + 3)$$
$$x = a$$

Substituting in (6), $\quad a^2 + a + 2y = a^2 + 3a$

$$2y = 2a$$
$$y = a$$

Substituting in (5), $\quad a + a + z = 2a + 1$

$$z = 1$$

30. $\begin{cases} x - ay + a^2z = a^3 & (1) \\ x - by + b^2z = b^3 & (2) \\ x - cy + c^2z = c^3 & (3) \end{cases}$

Subtracting (2) from (1), and (3) from (2),

$$-(a - b)y + (a^2 - b^2)z = a^3 - b^3 \quad (4)$$
$$-(b - c)y + (b^2 - c^2)z = b^3 - c^3 \quad (5)$$

Dividing (4) by $a - b$, and (5) by $b - c$,

$$-y + (a + b)z = a^2 + ab + b^2 \quad (6)$$
$$-y + (b + c)z = b^2 + bc + c^2$$

Subtracting, $\quad (a - c)z = a^2 - c^2 + b(a - c)$

Dividing by $a - c$, $\quad z = a + c + b$

Substituting in (6),

$$-y + a^2 + 2ab + b^2 + ac + bc = a^2 + ab + b^2$$
$$\therefore y = ab + bc + ca$$

Substituting in (1),

$$x - a^2b - abc - a^2c + a^3 + a^2c + a^2b = a^3$$
$$\therefore x = abc$$

CHAPTER XVI.

Art. 190. — Pages 149–157.

3. Let $x =$ the greater part,
and $y =$ the less.

By the conditions,
$$\begin{cases} x + y = 50 & (1) \\ \dfrac{3x}{8} = \dfrac{2y}{3} & (2) \end{cases}$$

From (2), $9x = 16y$, or $9x - 16y = 0$ (3)
Multiplying (1) by 9, $9x + 9y = 450$ (4)
Subtracting (3) from (4), $25y = 450$
Whence, $y = 18$, the less part.
Substituting in (1), $x + 18 = 50$
Whence, $x = 32$, the greater part.

4. Let $x =$ the greater number,
and $y =$ the less.

By the conditions,
$$\begin{cases} 7x = \dfrac{y}{7} + 97 & (1) \\ 7y = \dfrac{x}{7} + 47 & (2) \end{cases}$$

From (1), $49x = y + 679$, or $49x - y = 679$ (3)
From (2), $49y = x + 329$, or $49y - x = 329$ (4)
Subtracting (4) from (3), $50x - 50y = 350$
Or, $x - y = 7$ (5)
Subtracting (5) from (3), $48x = 672$
Whence, $x = 14$, the greater number.
Adding (4) and (5), $48y = 336$
Whence, $y = 7$, the less number.

5. Let $x =$ A's age,
and $y =$ B's age.

By the conditions,
$$\begin{cases} \dfrac{x}{5} + \dfrac{2y}{3} = \dfrac{58}{3} & (1) \\ \dfrac{7x}{8} - \dfrac{2y}{5} = \dfrac{73}{4} & (2) \end{cases}$$

From (1), $3x + 10y = 290$ (3)
From (2), $35x - 16y = 730$ (4)

Multiplying (3) by 8, $24x + 80y = 2320$

Multiplying (4) by 5, $175x - 80y = 3650$

Adding, $199x = 5970$

Whence, $x = 30$, A's age.

Substituting in (3), $90 + 10y = 290$

$10y = 200$

Whence, $y = 20$, B's age.

6. Let $x =$ the numerator,

and $y =$ the denominator.

By the conditions,
$$\begin{cases} \dfrac{x+1}{y} = \dfrac{1}{3} & (1) \\[2ex] \dfrac{x}{y+1} = \dfrac{1}{4} & (2) \end{cases}$$

From (1), $3x + 3 = y$, or $3x - y = -3$ (3)

From (2), $4x = y + 1$, or $4x - y = 1$ (4)

Subtracting (3) from (4), $x = 4$, the numerator.

Substituting in (4), $16 - y = 1$

Whence, $y = 15$, the denominator.

Therefore the fraction is $\frac{4}{15}$.

7. Let $x =$ the gentleman's age,

and $y =$ the wife's age.

After they had been married 12 years, his age was $x + 12$ years, and her age was $y + 12$ years. Hence by the conditions,

$$\begin{cases} y = \dfrac{3x}{4} & (1) \\[2ex] y + 12 = \dfrac{5}{6}(x + 12) & (2) \end{cases}$$

Substituting from (1) in (2), $\dfrac{3x}{4} + 12 = \dfrac{5x + 60}{6}$

$9x + 144 = 10x + 120$

$-x = -24$

Whence, $x = 24$, the gentleman's age,

and $y = \dfrac{3x}{4} = 18$, the wife's age.

8. Let $x =$ what A lost,

and $y =$ what B lost.

After settling their accounts, A had $240 - x$ dollars, and B had $96 - y$ dollars. Hence by the conditions,

$$\begin{cases} x = 2y & (1) \\ 240 - x = 3(96 - y) & (2) \end{cases}$$

Adding (1) and (2), $240 = 2y + 288 - 3y$
Whence, $y = 48$, what B lost.
Substituting in (1), $x = 96$, what A lost.

9. Let $x =$ A's age at present,
and $y =$ B's age at present.

By the conditions,
$$\begin{cases} x - 8 = 4(y - 8) & (1) \\ x + 12 = 2(y + 12) & (2) \end{cases}$$

From (1), $x - 8 = 4y - 32$, or $x - 4y = -24$ (3)
From (2), $x + 12 = 2y + 24$, or $x - 2y = 12$ (4)
Subtracting (3) from (4), $2y = 36$
Whence, $y = 18$, B's age at present.
Substituting in (4), $x - 36 = 12$
Whence, $x = 48$, A's age at present.

10. Let $x =$ the numerator,
and $y =$ the denominator.

By the conditions,
$$\begin{cases} \dfrac{x+5}{y+5} = \dfrac{1}{2} & (1) \\[2ex] \dfrac{x-3}{y-3} = \dfrac{1}{4} & (2) \end{cases}$$

From (1), $2x + 10 = y + 5$, or $2x - y = -5$ (3)
From (2), $4x - 12 = y - 3$, or $4x - y = 9$ (4)
Subtracting (3) from (4), $2x = 14$
Whence, $x = 7$, the numerator.
Substituting in (3), $14 - y = -5$
Whence, $y = 19$, the denominator.
Therefore the fraction is $\frac{7}{19}$.

11. Let $x =$ the number of days A requires,
and $y =$ the number of days B requires.
Then, $\dfrac{1}{x} =$ the part A does in one day,
and $\dfrac{1}{y} =$ the part B does in one day.
Also, $\dfrac{1}{10} =$ the part both together do in one day.

In 4 days A does $\dfrac{4}{x}$, and in 20 days B does $\dfrac{20}{y}$. Hence by the conditions,

$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{10} & (1) \\[2ex] \dfrac{4}{x} + \dfrac{20}{y} = 1 & (2) \end{cases}$$

Multiplying (1) by 4, $\frac{4}{x}+\frac{4}{y}=\frac{2}{5}$ (3)

Subtracting (3) from (2), $\frac{16}{y}=\frac{3}{5}$

$3y=80$

Whence, $y=\frac{80}{3}=26\frac{2}{3}$, the no. of days B requires.

Substituting in (1), $\frac{1}{x}+\frac{3}{80}=\frac{1}{10}$

$\frac{1}{x}=\frac{1}{16}$

Whence, $x=16$, the no. of days A requires.

12. Let $x=$ the greater number,

and $y=$ the less.

By the conditions, $\begin{cases} \frac{x}{y}=2+\frac{12}{y} & (1) \\ \frac{4y}{x}=1+\frac{14}{x} & (2) \end{cases}$

From (1), $x=2y+12$, or $x-2y=12$ (3)

From (2), $4y=x+14$, or $4y-x=14$ (4)

Adding (3) and (4), $2y=26$

Whence, $y=13$, the less number.

Substituting in (3), $x=26+12$

$=38$, the greater number.

13. Let $x=$ the numerator,

and $y=$ the denominator.

By the conditions, $\begin{cases} \frac{2x}{y+7}=\frac{2}{3} & (1) \\ \frac{x+2}{2y}=\frac{3}{5} & (2) \end{cases}$

From (1), $6x=2y+14$, or $3x-y=7$ (3)

From (2), $5x+10=6y$, or $5x-6y=-10$ (4)

Multiplying (3) by 6, $18x-6y=42$ (5)

Subtracting (4) from (5), $13x=52$

Whence, $x=4$, the numerator.

Substituting in (3), $12-y=7$

Whence, $y=5$, the denominator.

Therefore the fraction is $\frac{4}{5}$.

14. Let $x =$ the numerator,

and $y =$ the denominator.

By the conditions,

$$\begin{cases} \dfrac{x-(a-1)}{y} = a+1 & (1) \\[2ex] \dfrac{x}{y+a} = a & (2) \end{cases}$$

From (1), $x - a + 1 = ay + y$, or $x - ay - y = a - 1$ (3)

From (2), $x = ay + a^2$, or $x - ay \quad = a^2$ (4)

Subtracting (3) from (4), $y = a^2 - a + 1$, the denominator.

Substituting in (4),

$$x = a(a^2 - a + 1) + a^2$$
$$= a^3 - a^2 + a + a^2$$
$$= a^3 + a, \text{ the numerator.}$$

Therefore the fraction is $\dfrac{a^3 + a}{a^2 - a + 1}$.

15. Let $x =$ the value of the better horse,

$y =$ the value of the poorer horse,

and $z =$ the value of the harness.

By the conditions,
$$\begin{cases} x + y + z = 300 & (1) \\ y + z = x - 20 & (2) \\ x + z = 2y & (3) \end{cases}$$

Subtracting (2) from (1), $x = 320 - x$

$2x = 320$

Whence, $x = 160$, the value of the better horse.

Subtracting (3) from (1), $y = 300 - 2y$

$3y = 300$

Whence, $y = 100$, the value of the poorer horse.

Substituting in (3), $160 + z = 200$

Whence, $z = 40$, the value of the harness.

16. Let $x =$ the price per lb. of the 1st quality,

$y =$ the price per lb. of the 2d,

and $z =$ the price per lb. of the 3d.

By the conditions,

$$\begin{cases} 3x + 4y + 2z = 60 & (1) \\ 4x + y + 5z = 59 & (2) \\ x + 10y + 3z = 90 & (3) \end{cases}$$

Multiplying (3) by 3,

$$3x + 30y + 9z = 270$$
$$3x + 4y + 2z = 60 \qquad (1)$$

Subtracting, $26y + 7z = 210$ (4)

Multiplying (3) by 4,

$$4x + 40y + 12z = 360$$
$$4x + y + 5z = 59 \quad (2)$$

Subtracting, $39y + 7z = 301$ (5)

$$26y + 7z = 210 \quad (4)$$

Subtracting, $13y = 91$

Whence, $y = 7$, the price per lb. of the 2d quality.

Substituting in (4), $182 + 7z = 210$

$$7z = 28$$

Whence, $z = 4$, the price per lb. of the 3d quality.

Substituting in (3),

$$x + 70 + 12 = 90$$

Whence, $x = 8$, the price per lb. of the 1st quality.

18. Let $x =$ the number of oranges,
and $y =$ the amount paid for each.
Then, $xy =$ the amount of money spent.
By the conditions,

$$\begin{cases} (x+5)\left(y - \frac{1}{2}\right) = xy & (1) \\ (x-3)\left(y + \frac{1}{2}\right) = xy & (2) \end{cases}$$

From (1), $xy - \frac{x}{2} + 5y - \frac{5}{2} = xy$, or $-\frac{x}{2} + 5y = \frac{5}{2}$ (3)

From (2), $xy + \frac{x}{2} - 3y - \frac{3}{2} = xy$, or $\frac{x}{2} - 3y = \frac{3}{2}$ (4)

Adding (3) and (4), $2y = 4$

$$y = 2$$

Substituting in (4), $\frac{x}{2} - 6 = \frac{3}{2}$

$$x - 12 = 3$$

Whence, $x = 15$, the number of oranges,
and $xy = 30$, the amount of money spent.

19. Let $x =$ the number of bushels at 60 cents,
and $y =$ the number at 90 cents.

By the conditions, $\begin{cases} x + y = 40 & (1) \\ 60x + 90y = 3200 & (2) \end{cases}$

Multiplying (1) by 6, $6x + 6y = 240$ (3)

Dividing (2) by 10, $6x + 9y = 320$ (4)

Subtracting (3) from (4), $3y = 80$

Whence, $y = 26\frac{2}{3}$, the no. of bushels at 90 cents.

Substituting in (1), $x + 26\frac{2}{3} = 40$

Whence, $x = 13\frac{1}{3}$, the no. of bushels at 60 cents.

20. Let $x =$ the income tax,

and $y =$ the assessed tax.

By the conditions,
$$\begin{cases} x + y = 50 & (1) \\ \dfrac{3x}{2} + \dfrac{3y}{4} = 52.50 & (2) \end{cases}$$

Multiplying (2) by $\frac{4}{3}$, $2x + y = 70$ (3)

Subtracting (1) from (3), $x = 20$, the income tax.

Substituting in (1), $20 + y = 50$

Whence, $y = 30$, the assessed tax.

21. Let $x =$ the number of eggs bought,

and $y =$ the amount paid for each.

Then, $xy =$ the amount of money spent.

By the conditions,
$$\begin{cases} (x + 20)(y - 1) = xy & (1) \\ (x - 15)(y + 1) = xy & (2) \end{cases}$$

From (1), $xy - x + 20y - 20 = xy$, or $-x + 20y = 20$ (3)

From (2), $xy + x - 15y - 15 = xy$, or $x - 15y = 15$ (4)

Adding (3) and (4), $5y = 35$

Whence, $y = 7$, the amount paid for each egg.

Substituting in (4), $x - 105 = 15$

Whence, $x = 120$, the number of eggs bought.

22. Let $x =$ the length in feet,

and $y =$ the width.

Then, $xy =$ the area in square feet.

By the conditions,
$$\begin{cases} (x + 8)(y + 2) = xy + 656 & (1) \\ (x + 2)(y + 8) = xy + 776 & (2) \end{cases}$$

From (1), $xy + 2x + 8y + 16 = xy + 656$, or $x + 4y = 320$ (3)

From (2), $xy + 8x + 2y + 16 = xy + 776$, or $4x + y = 380$ (4)

Multiplying (3) by 4, $4x + 16y = 1280$ (5)

Subtracting (4) from (5), $15y = 900$

Whence, $y = 60$, the width in feet.

Substituting in (3), $x + 240 = 320$

Whence, $x = 80$, the length.

23. Let $x =$ the amount A has,
and $y =$ the amount B has.

By the conditions,

$$\begin{cases} x + 5 = y - 5 & (1) \\ y + 15 = \frac{7}{3}(x - 15) & (2) \end{cases}$$

From (1), $x - y = -10$ (3)

From (2), $3y + 45 = 7x - 105$

or, $3y - 7x = -150$ (4)

Multiplying (3) by 3, $3x - 3y = -30$ (5)

Adding (4) and (5), $-4x = -180$

Whence, $x = 45$, the amount A has.

Substituting in (3), $45 - y = -10$

$-y = -55$

Whence, $y = 55$, the amount B has.

24. Let the numbers be x, y, and z; then by the conditions,

$$\begin{cases} x + \frac{y}{2} + \frac{z}{2} = 34 & (1) \\ \frac{x}{3} + y + \frac{z}{3} = 34 & (2) \\ \frac{x}{4} + \frac{y}{4} + z = 34 & (3) \end{cases}$$

From (1), $2x + y + z = 68$ (4)

From (2), $x + 3y + z = 102$ (5)

From (3), $x + y + 4z = 136$ (6)

Subtracting (6) from (5), $2y - 3z = -34$ (7)

Multiplying (5) by 2, $2x + 6y + 2z = 204$

$2x + y + z = 68$ (4)

Subtracting, $5y + z = 136$ (8)

Multiplying (8) by 3, $15y + 3z = 408$

$2y - 3z = -34$ (7)

Adding, $17y = 374$

Whence, $y = 22$

Substituting in (7), $44 - 3z = -34$

$3z = 78$

Whence, $z = 26$

Substituting in (1), $x + 11 + 13 = 34$

Whence, $x = 10$

Therefore the numbers are 10, 22, and 26.

25. Let the numbers be u, x, y, and z; then by the conditions,

$$\begin{cases} u+x+y+z=136 & (1) \\ 2u-x=\ 46 & (2) \\ 2x-y=\ 44 & (3) \\ 2y-z=\ 40 & (4) \end{cases}$$

From (2), $x=2u-\ 46$ (5)

Substituting in (3), $4u-92-y=44$, or $y=4u-136$ (6)

Substituting in (4), $8u-272-z=40$, or $z=8u-312$ (7)

Substituting from (5), (6), and (7) in (1),

$$u+2u-46+4u-136+8u-312=136$$

$$15u=630$$

Whence, $u=\ 42$

Substituting in (5), $x=\ 38$

Substituting in (6), $y=\ 32$

Substituting in (7), $z=\ 24$

Therefore the numbers are 42, 38, 32, and 24.

27. Let $x=$ the first digit,

and $y=$ the second.

Then, $10x+y=$ the number,

and $10y+x=$ the number with its digits inverted.

By the conditions,

$$\begin{cases} x+y=11 & (1) \\ 10x+y-27=10y+x & (2) \end{cases}$$

From (2), $9x-9y=27$, or $x-y=3$ (3)

Adding (1) and (3), $2x=14$

Whence, $x=\ 7$, the first digit.

Subtracting (3) from (1), $2y=\ 8$

Whence, $y=\ 4$, the second digit.

Therefore the number is 74.

28. Let $x=$ the first digit,

and $y=$ the second.

Then, $2x=$ the third digit.

Also, $100x+10y+2x$

or $102x+10y=$ the number,

and $200x+10y+x$

or $201x+10y=$ the number with its digits inverted.

By the conditions.

$$\begin{cases} 3x+y=11 & (1) \\ 102x+10y+297=201x+10y & (2) \end{cases}$$

From (2), $297=99x$

Whence, $x = 3$, the first digit.

Substituting in (1), $9 + y = 11$

Whence, $y = 2$, the second digit.

Also, $2x = 6$, the third digit.

Therefore the number is 326.

29. Let $x =$ the number of days A requires,

$y =$ the number B requires,

and $z =$ the number C requires.

Then, $\frac{1}{x} =$ the part A does in one day,

$\frac{1}{y} =$ the part B does in one day,

and $\frac{1}{z} =$ the part C does in one day.

Also, $\frac{1}{6} =$ the part A and B together do in one day,

$\frac{1}{8} =$ the part A and C together do in one day,

and $\frac{1}{12} =$ the part B and C together do in one day.

By the conditions,
$$\begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{1}{6} & (1) \\ \frac{1}{x} + \frac{1}{z} = \frac{1}{8} & (2) \\ \frac{1}{y} + \frac{1}{z} = \frac{1}{12} & (3) \end{cases}$$

Adding (1), (2), and (3),
$$\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = \frac{3}{8}$$

Or,
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{16} \qquad (4)$$

Subtracting (3) from (4), $\frac{1}{x} = \frac{3}{16} - \frac{1}{12} = \frac{5}{48}$

Whence, $x = \frac{48}{5} = 9\frac{3}{5}$, the number of days A requires.

Subtracting (2) from (4), $\frac{1}{y} = \frac{3}{16} - \frac{1}{8} = \frac{1}{16}$

Whence, $y = 16$, the number of days B requires.

Subtracting (1) from (4), $\frac{1}{z} = \frac{3}{16} - \frac{1}{6} = \frac{1}{48}$

Whence, $z = 48$, the number of days C requires.

30. Let $x =$ the length in rods,

and $y =$ the width.

Then, $xy =$ the area in square rods.

By the conditions, $\begin{cases} (x+5)(y+4) = xy + 240 & (1) \\ (x-4)(y-5) = xy - 210 & (2) \end{cases}$

From (1), $xy + 4x + 5y + 20 = xy + 240$, or $4x + 5y = 220$ (3)

From (2), $xy - 5x - 4y + 20 = xy - 210$, or $5x + 4y = 230$ (4)

Multiplying (3) by 4, $16x + 20y = 880$ (5)

Multiplying (4) by 5, $25x + 20y = 1150$ (6)

Subtracting (5) from (6), $9x = 270$

Whence, $x = 30$, the length in rods.

Substituting in (3), $120 + 5y = 220$

$5y = 100$

Whence, $y = 20$, the width in rods.

Therefore, $xy = 600$, the area in square rods.

31. Let x, y, and z represent the numbers; then by the conditions,

$$\begin{cases} x + y = c & (1) \\ y + z = a & (2) \\ z + x = b & (3) \end{cases}$$

Adding, $2x + 2y + 2z = a + b + c$ (4)

Subtracting twice (2) from (4), $2x = b + c - a$

Whence, $x = \dfrac{b + c - a}{2}$

Subtracting twice (3) from (4), $2y = c + a - b$

Whence, $y = \dfrac{c + a - b}{2}$

Subtracting twice (1) from (4), $2z = a + b - c$

Whence, $z = \dfrac{a + b - c}{2}$

32. Let $x =$ the first digit,

$y =$ the second,

and $z =$ the third.

Then, $100x + 10y + z =$ the number,

and $100z + 10y + x =$ the no. with its digits inverted.

By the conditions,

$$\begin{cases} x - y = y - z & (1) \\ \dfrac{100x + 10y + z}{\frac{1}{2}(x + y + z)} = 41 & (2) \\ 100x + 10y + z + 396 = 100z + 10y + x & (3) \end{cases}$$

From (1), $x - 2y + z = 0$ (4)

From (2), $200x + 20y + 2z = 41x + 41y + 41z$

Or, $159x - 21y - 39z = 0$ (5)

Dividing (5) by 3, $53x - 7y - 13z = 0$ (6)

From (3), $99x = 99z - 396$, or $x = z - 4$ (7)

Substituting in (4), $z - 4 - 2y + z = 0$

$$2z - 4 = 2y$$

Or, $y = z - 2$ (8)

Substituting from (7) and (8) in (6),

$$53(z-4) - 7(z-2) - 13z = 0$$

$$53z - 212 - 7z + 14 - 13z = 0$$

$$33z = 198$$

Whence, $z = 6$, the third digit.

Substituting in (7) and (8), $x = 2$, the first digit,

and $y = 4$, the second.

Therefore the number is 246.

33. Let $x =$ the number of persons,

and $y =$ what each received.

Then, $xy =$ the sum divided.

By the conditions,

$$\begin{cases} (x+m)(y-a) = xy & (1) \\ (x-n)(y+b) = xy & (2) \end{cases}$$

From (1), $xy - ax + my - am = xy$, or $-ax + my = am$ (3)

From (2), $xy + bx - ny - bn = xy$, or $bx - ny = bn$ (4)

Multiplying (3) by n, $-anx + mny = amn$

Multiplying (4) by m, $bmx - mny = bmn$

Adding, $(bm - an)x = amn + bmn$

Whence, $x = \dfrac{(a+b)mn}{bm - an}$, the number of persons.

Multiplying (3) by b, $-abx + bmy = abm$

Multiplying (4) by a, $abx - any = abn$

Adding, $(bm - an)y = abm + abn$

Whence, $y = \dfrac{ab(m+n)}{bm - an}$, what each received.

34. Let $u =$ the share of the eldest,

$x =$ the share of the second,

$y =$ the share of the third,

and $z =$ the share of the youngest.

By the conditions,
$$\begin{cases} u = \dfrac{x+y+z}{2} & (1) \\ x = \dfrac{u+y+z}{3} & (2) \\ y = \dfrac{u+x+z}{4} & (3) \\ u = z + 140 & (4) \end{cases}$$

From (1), $2u - x - y - z = 0$ (5)
From (2), $-u + 3x - y - z = 0$ (6)
From (3), $-u - x + 4y - z = 0$ (7)
From (4), $u - z = 140$ (8)
Subtracting (6) from (5), $3u - 4x = 0$ (9)
Subtracting (7) from (5), $3u - 5y = 0$ (10)
Subtracting (8) from (5), $u - x - y = -140$ (11)
From (9), $x = \dfrac{3u}{4}$ (12)
From (10), $y = \dfrac{3u}{5}$ (13)

Substituting from (12) and (13) in (11),
$$u - \frac{3u}{4} - \frac{3u}{5} = -140$$
$$20u - 15u - 12u = -2800$$
$$-7u = -2800$$

Whence, $u = 400$, the share of the eldest.
Substituting in (12), $x = 300$, the share of the second.
Substituting in (13), $y = 240$, the share of the third.
Substituting in (8), $400 - z = 140$
Whence, $z = 260$, the share of the youngest.
And $u + x + y + z = 1200$, the whole sum divided.

35. Let $x =$ the number at 2 for 5 cents,
and $y =$ the number at 3 for 3 cents.

By the conditions,
$$\begin{cases} \dfrac{5x}{2} + \dfrac{3y}{3} = 171 & (1) \\ 3x + 3y = 171 + 27 = 198 & (2) \end{cases}$$

From (1), $15x + 16y = 1026$
Multiplying (2) by 5, $15x + 15y = 990$

Subtracting, $y = 36$, the number at 3 for 3 cents.
Substituting in (2), $3x + 108 = 198$
$3x = 90$
Whence, $x = 30$, the number at 2 for 5 cents.

36. Let $x =$ the first digit,

and $y =$ the second.

Then, $10x + y =$ the number,

and $10y + x =$ the no. with its digits inverted.

By the conditions,
$$\begin{cases} \dfrac{10x+y}{x+y} = 7 & (1) \\[2ex] \dfrac{10y+x+6}{x+y} = 5 & (2) \end{cases}$$

From (1), $10x + y = 7x + 7y$, or $x = 2y$ (3)

From (2), $10y + x + 6 = 5x + 5y$, or $5y - 4x = -6$ (4)

Substituting from (3) in (4),

$$5y - 8y = -6$$
$$-3y = -6$$

Whence, $y = 2$, the second digit.

Substituting in (3), $x = 4$, the first digit.

Therefore the number is 42.

37. Let $x =$ the first digit,

and $y =$ the second.

Then, $10x + y =$ the number,

and $10y + x =$ the no. with its digits inverted.

By the conditions,
$$\begin{cases} 10x + y + 45 = 10y + x & (1) \\[1ex] \dfrac{10y+x}{x+y} = 7 + \dfrac{6}{x+y} & (2) \end{cases}$$

From (1), $9x - 9y = -45$, or $x - y = -5$ (3)

From (2), $10y + x = 7x + 7y + 6$, or $y - 2x = 2$ (4)

Adding (3) and (4), $-x = -3$

Whence, $x = 3$, the first digit.

Substituting in (4), $y - 6 = 2$

Whence, $y = 8$, the second digit.

Therefore the number is 38.

38. Let $x =$ the number of days A requires,

$y =$ the number B requires,

and $z =$ the number C requires.

Then, $\frac{1}{x} =$ the part A does in one day,

$\frac{1}{y} =$ the part B does in one day,

and $\frac{1}{z} =$ the part C does in one day.

Also, $\frac{1}{m}=$ the part A and B together do in one day,

$\frac{1}{n}=$ the part B and C together do in one day,

and $\frac{1}{p}=$ the part C and A together do in one day.

By the conditions,
$$\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{1}{m} & (1) \\ \frac{1}{y}+\frac{1}{z}=\frac{1}{n} & (2) \\ \frac{1}{z}+\frac{1}{x}=\frac{1}{p} & (3) \end{cases}$$

Adding, $$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{m}+\frac{1}{n}+\frac{1}{p} \quad (4)$$

Subtracting twice (2) from (4),
$$\frac{2}{x}=\frac{1}{p}+\frac{1}{m}-\frac{1}{n}=\frac{mn+np-mp}{mnp}$$

Whence, $x=\frac{2mnp}{mn+np-mp}$, the no. of days A requires.

Subtracting twice (3) from (4),
$$\frac{2}{y}=\frac{1}{n}+\frac{1}{m}-\frac{1}{p}=\frac{mp+np-mn}{mnp}$$

Whence, $y=\frac{2mnp}{mp+np-mn}$, the no. of days B requires.

Subtracting twice (1) from (4),
$$\frac{2}{z}=\frac{1}{n}+\frac{1}{p}-\frac{1}{m}=\frac{mp+mn-np}{mnp}$$

Whence, $z=\frac{2mnp}{mp+mn-np}$, the no. of days C requires.

40. Let $x=$ the rate of the crew in still water,
and $y=$ the rate of the current.
Then, $x+y=$ the rate rowing down stream,
and $x-y=$ the rate rowing up stream.

By the conditions,
$$\begin{cases} \frac{a}{x+y}=b & (1) \\ \frac{c}{x-y}=d & (2) \end{cases}$$

From (1), $bx+by=a$ (3)

From (2), $dx-dy=c$ (4)

Multiplying (3) by d,
$$bdx+bdy=ad \quad (5)$$

Multiplying (4) by b,

$$bdx - bdy = bc \qquad (6)$$

Adding (5) and (6), $2\,bdx = ad + bc$

Whence, $x = \dfrac{ad + bc}{2\,bd}$, the rate of the crew.

Subtracting (6) from (5),

$$2\,bdy = ad - bc$$

Whence, $y = \dfrac{ad - bc}{2\,bd}$, the rate of the current.

41. Let $x =$ the boatman's rate rowing down stream,
and $y =$ his rate rowing up stream.

Then, $\dfrac{20}{x} =$ his time in going,

and $\dfrac{20}{y} =$ his time in returning.

By the conditions,

$$\begin{cases} \dfrac{20}{x} + \dfrac{20}{y} = 10 & (1) \\ \dfrac{2}{y} = \dfrac{3}{x} & (2) \end{cases}$$

From (1), $\dfrac{2}{x} + \dfrac{2}{y} = 1 \qquad (3)$

From (2), $\dfrac{2}{y} - \dfrac{3}{x} = 0 \qquad (4)$

Subtracting (4) from (3), $\dfrac{5}{x} = 1$

Multiplying by (4), $\dfrac{20}{x} = 4$, the boatman's time in going.

Substituting in (1), $4 + \dfrac{20}{y} = 10$

Whence, $\dfrac{20}{y} = 6$, his time in returning.

42. Let $x =$ the first digit,
$y =$ the second,
and $z =$ the third.

Then, $100\,x + 10\,y + z =$ the number,
and $100\,z + 10\,y + x =$ the number with its digits inverted.

By the conditions,

$$\begin{cases} x + y + z = 21 & (1) \\ x + 2\,y - z = 8 & (2) \\ 100\,x + 10\,y + z + 198 = 100\,z + 10\,y + x & (3) \end{cases}$$

From (3), $99x - 99z = -198$, or $x - z = -2$ (4)

Adding (1) and (2), $2x + 3y = 29$

Adding (1) and (4), $2x + y = 19$ (5)

Subtracting, $2y = 10$

Whence, $y = 5$, the second digit.

Substituting in (5), $2x + 5 = 19$

$2x = 14$

Whence, $x = 7$, the first digit.

Substituting in (4), $7 - z = -2$

Whence, $z = 9$, the third digit.

Therefore the number is 759.

43. Let $x =$ the number of gallons in the first cask,

and $y =$ the number in the second.

After the first pouring, the first cask contains $x - y$ gallons, and the second $2y$ gallons.

After the second pouring, the first contains $x - y + x - y$, or $2x - 2y$ gallons, and the second contains $2y - (x - y)$, or $3y - x$ gallons.

After the third pouring, the first contains $2x - 2y - (3y - x)$, or $3x - 5y$ gallons, and the second contains $2(3y - x)$, or $6y - 2x$ gallons.

Then by the conditions,

$$\begin{cases} 3x - 5y = 16 & (1) \\ 6y - 2x = 16 & (2) \end{cases}$$

Multiplying (2) by $\frac{3}{2}$,

$$9y - 3x = 24 \quad (3)$$

Adding (1) and (3), $4y = 40$

Whence, $y = 10$, the no. of gallons in the second cask.

Substituting in (1), $3x - 50 = 16$

Whence, $x = 22$, the no. in the first cask.

44. Let $x =$ the first digit,

and $y =$ the second.

Then, $10x + y =$ the number,

and $10y + x =$ the number with its digits inverted.

By the conditions,

$$\begin{cases} 10x + y + 10y + x = 121 & (1) \\ \dfrac{10x + y}{x + y} = 5 + \dfrac{10}{x + y} & (2) \end{cases}$$

From (1), $11x + 11y = 121$, or $x + y = 11$ (3)

From (2), $10x + y = 5x + 5y + 10$, or $5x - 4y = 10$ (4)

Multiplying (3) by 4, $4x + 4y = 44$ (5)

Adding (4) and (5), $9x = 54$

Whence, $x = 6$, the first digit.

Substituting in (3), $6 + y = 11$

Whence, $y = 5$, the second digit.

Therefore the number is 65.

45. Let $x =$ the first rate,

and $y =$ the second.

By the conditions,

$$\begin{cases} 300x - 200y = 800 & (1) \\ 350y - 240x = 310 & (2) \end{cases}$$

From (1), $3x - 2y = 8$ (3)

From (2), $35y - 24x = 31$ (4)

Multiplying (3) by 8, $24x - 16y = 64$ (5)

Adding (4) and (5), $19y = 95$

Whence, $y = 5$, the second rate.

Substituting in (3), $3x - 10 = 8$

Whence, $x = 6$, the first rate.

46. Let $x =$ the sum,

and $y =$ the rate per cent.

By the conditions, $$\begin{cases} x + \dfrac{2xy}{100} = 132 & (1) \\ x + \dfrac{5xy}{100} = 150 & (2) \end{cases}$$

Subtracting (1) from (2), $\dfrac{3xy}{100} = 18$

Or, $\dfrac{xy}{100} = 6$ (3)

Substituting in (1), $x + 12 = 132$

Whence, $x = 120$, the sum let.

Substituting in (3), $\dfrac{120y}{100} = 6$

Whence, $y = 5$, the rate per cent.

47. Let $x =$ the sum,

and $y =$ the rate per cent.

By the conditions, $$\begin{cases} x + \dfrac{mxy}{100} = a & (1) \\ x + \dfrac{nxy}{100} = b & (2) \end{cases}$$

Subtracting (2) from (1),

$$\frac{(m-n)xy}{100}=a-b$$

Or, $$\frac{xy}{100}=\frac{a-b}{m-n} \qquad (3)$$

Substituting in (1),

$$x+\frac{m(a-b)}{m-n}=a$$

$$(m-n)\,x+am-bm=am-an$$

Whence, $$x=\frac{bm-an}{m-n}, \text{ the sum let.}$$

Substituting in (3),

$$\frac{bm-an}{m-n}\cdot\frac{y}{100}=\frac{a-b}{m-n}$$

Whence, $$y=\frac{100\,(a-b)}{bm-an}, \text{ the rate per cent.}$$

49. Let $x=$ the distance rowed down stream,

and $y=$ his rate of rowing in miles per hour.

Then, $y+\frac{7}{2}=$ his rate rowing down stream,

and $y-\frac{7}{2}=$ his rate rowing up stream.

By the conditions,
$$\begin{cases}\dfrac{x}{y+\frac{7}{2}}=\dfrac{5}{3} & (1)\\[2ex] \dfrac{x-2}{y-\frac{7}{2}}=6\tfrac{1}{2}=\dfrac{13}{2} & (2)\end{cases}$$

From (1), $$3\,x=5\,y+\frac{35}{2}, \text{ or } 6\,x-10\,y=35 \qquad (3)$$

From (2), $$2\,x-4=13\,y-\frac{91}{2}, \text{ or } 4\,x-26\,y=-83 \qquad (4)$$

Multiplying (3) by 2, $12x-20y=70$

Multiplying (4) by 3, $12x-78y=-249$

Subtracting, $58\,y=319$

Whence, $y=\frac{319}{58}=5\frac{1}{2}$, his rate of rowing.

Substituting in (3), $6\,x-55=35$

Whence, $x=15$, the distance rowed down stream.

50. Let $x =$ the first digit,
and $y =$ the second.
Then, $10x + y =$ the number,
and $10y + x =$ the number with its digits inverted.
By the conditions,

$$\begin{cases} \dfrac{10x + y}{x + y} = 6 + \dfrac{1}{x + y} & (1) \\ \dfrac{10y + x + 8}{x + y} = 6 & (2) \end{cases}$$

From (1), $10x + y = 6x + 6y + 1$, or $4x - 5y = 1$ (3)
From (2), $10y + x + 8 = 6x + 6y$, or $4y - 5x = -8$ (4)
Multiplying (3) by 5, $20x - 25y = 5$
Multiplying (4) by 4, $16y - 20x = -32$
Adding, $-9y = -27$
Whence, $y = 3$, the second digit.
Substituting in (3), $4x - 15 = 1$
Whence, $x = 4$, the first digit.
Therefore the number is 43.

51. Let $5x =$ the rate of the train before the accident.
Then, $3x =$ its rate afterwards.
Let $y =$ the dis. to B from the point of detention.

By the conditions,
$$\begin{cases} \dfrac{1}{2} + \dfrac{y}{3x} = \dfrac{y}{5x} + \dfrac{5}{2}, \quad \text{or} \quad \dfrac{y}{3x} - \dfrac{y}{5x} = 2 & (1) \\ \dfrac{1}{2} + \dfrac{y + 30}{3x} = \dfrac{y + 30}{5x} + 3, \text{ or } \dfrac{y + 30}{3x} - \dfrac{y + 30}{5x} = \dfrac{5}{2} & (2) \end{cases}$$

Subtracting (1) from (2),
$$\frac{30}{3x} - \frac{30}{5x} = \frac{1}{2}$$
Or, $\dfrac{4}{x} = \dfrac{1}{2}$
Whence, $x = 8$
Therefore, $5x = 40$, the rate before the accident.

52. Let $x =$ what A had at first,
$y =$ what B had,
and $z =$ what C had.
If A gives to B and C as much as each of them has, A will have $x - y - z$ dollars, B $2y$ dollars, and C $2z$ dollars.

If B gives to A and C as much as each of them now has, A will have $2x-2y-2z$ dollars, B $2y-(x-y-z)-2z$, or $3y-x-z$ dollars, and C $4z$ dollars.

If C gives to A and B as much as each of them now has, A will have $4x-4y-4z$ dollars, B $6y-2x-2z$ dollars, and C $4z-(2x-2y-2z)-(3y-x-z)$, or $7z-x-y$ dollars.

Then by the conditions,

$$\begin{cases} x+y+z=24 & (1) \\ 4x-4y-4z=8 & (2) \\ 6y-2x-2z=8 & (3) \end{cases}$$

From (2), $x-y-z=2$ (4)

From (3), $3y-x-z=4$ (5)

Adding (1) and (4), $2x=26$

Whence, $x=13$, what A had at first.

Adding (1) and (5), $4y=28$

Whence, $y=7$, what B had at first.

Substituting in (1), $13+7+z=24$

Whence, $z=4$, what C had at first.

53. Let $x=$ the no. of feet A builds in 1 hour,

and $y=$ the no. of feet B builds.

By the conditions, $\begin{cases} 3x+17y=126 & (1) \\ 7x+11\frac{2}{3}y=126 & (2) \end{cases}$

From (2), $21x+35y=378$

Or, $3x+5y=54$ (3)

Subtracting (3) from (1), $12y=72$

Whence, $y=6$, the no. of feet B builds in 1 hour.

Substituting in (1), $3x+102=126$

Whence, $x=8$, the no. of feet A builds in 1 hour.

54. Let x, y, and z represent the parts; then by the conditions,

$$\begin{cases} x+y+z=115 & (1) \\ x+30=2y+2=6z+4 & (2) \end{cases}$$

From (2), $x+30=6z+4$, or $x=6z-26$ (3)

From (2), $2y+2=6z+4$, or $y=3z+1$ (4)

Substituting in (1),

$$6z-26+3z+1+z=115$$

$$10z=140$$

Whence, $z=14$

Substituting in (3), $x=58$

Substituting in (4), $y=43$

Therefore the parts are 58, 43, and 14.

55. Let $u =$ what A had at first,
$x =$ what B had,
$y =$ what C had,
and $z =$ what D had.

By the conditions,
$$\begin{cases} x - y = 1 & (1) \\ u + \frac{x}{2} = 18 & (2) \\ \frac{x}{2} + \frac{y}{3} = 18 & (3) \\ \frac{2y}{3} + \frac{z}{4} = 18 & (4) \end{cases}$$

From (2), $2u + x = 36$, or $x = 36 - 2u$ (5)

Subtracting (3) from (2), $u - \frac{y}{3} = 0$

Or, $y = 3u$ (6)

Substituting in (1), $36 - 2u - 3u = 1$

$5u = 35$

Whence, $u = 7$, what A had at first.

Substituting in (5), $x = 22$, what B had.

Substituting in (6), $y = 21$, what C had.

Substituting in (4), $14 + \frac{z}{4} = 18$

$\frac{z}{4} = 4$

Whence, $z = 16$, what D had.

56. Let $x =$ what A had at first,
$y =$ what B had,
and $z =$ what C had.

If A gives to B and C as much as each of them has, A will have $x - y - z$ dollars, B $2y$ dollars, and C $2z$ dollars.

If B gives to A and C as much as each of them now has, A will have $2x - 2y - 2z$ dollars, B $2y - (x - y - z) - 2z$, or $3y - x - z$ dollars, and C $4z$ dollars.

If C gives to A and B as much as each of them now has, A will have $4x - 4y - 4z$ dollars, B $6y - 2x - 2z$ dollars, and C $4z - (2x - 2y - 2z) - (3y - x - z)$, or $7z - x - y$ dollars.

Then by the conditions,
$$\begin{cases} 4x - 4y - 4z = 48 & (1) \\ 6y - 2x - 2z = 48 & (2) \\ 7z - x - y = 48 & (3) \end{cases}$$

From (1), $x - y - z = 12$ (4)

From (2), $3y - x - z = 24$ (5)

Adding (3) and (4), $6z - 2y = 60$

Or, $3z - y = 30$ (6)

Adding (4) and (5), $2y - 2z = 36$

Or, $y - z = 18$ (7)

Adding (6) and (7), $2z = 48$

Whence, $z = 24$, what C had at first.

Substituting in (7), $y - 24 = 18$

Whence, $y = 42$, what B had at first.

Substituting in (4), $x - 42 - 24 = 12$

Whence, $x = 78$, what A had at first.

57. Let $x =$ the no. of hrs. A takes to mow an acre,

$y =$ the no. of hours B takes,

and $z =$ the no. of hours C takes.

Then, $\frac{1}{x} =$ the part of an acre A mows in one hour,

$\frac{1}{y} =$ the part B mows in one hour,

and $\frac{1}{z} =$ the part C mows in one hour.

By the conditions,

$$\begin{cases} \frac{2}{x} + \frac{3}{y} + \frac{5}{z} = 1 & (1) \\ \frac{4}{x} + \frac{9}{y} + \frac{6}{z} = 2 & (2) \\ \frac{10}{x} + \frac{12}{y} + \frac{5}{z} = 3 & (3) \end{cases}$$

Subtracting (1) from (3),

$$\frac{8}{x} + \frac{9}{y} = 2 \qquad (4)$$

Multiplying (1) by 6,

$$\frac{12}{x} + \frac{18}{y} + \frac{30}{z} = 6 \qquad (5)$$

Multiplying (2) by 5.

$$\frac{20}{x} + \frac{45}{y} + \frac{30}{z} = 10 \qquad (6)$$

Subtracting (5) from (6),

$$\frac{8}{x} + \frac{27}{y} = 4 \qquad (7)$$

Subtracting (4) from (7),

$$\frac{18}{y}=2$$

Whence, $y=9$, the no. of hours B takes to mow an acre.

Substituting in (4),

$$\frac{8}{x}+1=2$$

Whence, $x=8$, the no. of hrs. A takes to mow an acre.

Substituting in (2),

$$\frac{1}{2}+1+\frac{6}{z}=2$$

$$\frac{6}{z}=\frac{1}{2}$$

Whence, $z=12$, the no. of hrs. C takes to mow an acre.

58. Let $100x=$ the amount in $3\frac{1}{2}$ per cent bonds,

and $100y=$ the amount in 4 per cent bonds.

Then, $\frac{3\frac{1}{2}}{100}\times 100x$, or $\frac{7x}{2}=$ the income from $3\frac{1}{2}$ per cents,

and

$\frac{4}{100}\times 100y$, or $4y=$ the income from the 4 per cents.

By the conditions,

$$\begin{cases} 100x+100y=3600, \text{ or } \quad x+\ \ y=36 & (1) \\ \frac{7x}{2}-\ \ 4y=\ \ \ \ 6, \text{ or } 7x-8y=12 & (2) \end{cases}$$

Multiplying (1) by 7,

$$7x+7y=252$$

From (2), $7x-8y=\ \ 12$

Subtracting, $15y=240$

Whence, $y=16$

Substituting in (1),

$$x+16=36$$

Whence, $x=20$

Therefore, $100x=2000$, the amount in $3\frac{1}{2}$ per cent bonds,

and $100y=1600$, the amount in 4 per cent bonds.

59. Let $x=$ the number of feet A runs in a second,

and $y=$ the number B runs.

The first heat, A runs 480 feet in 6 seconds less time than B runs 432 feet; the second heat, A runs 480 feet in 2 seconds more time than B runs 336 feet.

Then by the conditions,

$$\begin{cases} \dfrac{480}{x} + 6 = \dfrac{432}{y} & (1) \\ \dfrac{480}{x} - 2 = \dfrac{336}{y} \end{cases}$$

Subtracting, $8 = \dfrac{96}{y}$

Whence, $y = 12$, the no. of feet B runs in a second.

Substituting in (1),

$$\frac{480}{x} + 6 = 36$$

Whence, $x = 16$, the no. of feet A runs in a second.

60. Let $x =$ the circumference of the hind-wheel in feet,

and $y =$ the circumference of the fore-wheel.

By the conditions,
$$\begin{cases} \dfrac{96}{y} = \dfrac{96}{x} + 4, \text{ or } \dfrac{24}{y} - \dfrac{24}{x} = 1 & (1) \\ \dfrac{96}{\frac{3y}{2}} = \dfrac{96}{\frac{4x}{3}} + 2, \text{ or } \dfrac{64}{y} - \dfrac{72}{x} = 2 & (2) \end{cases}$$

Multiplying (1) by 3,

$$\frac{72}{y} - \frac{72}{x} = 3 \qquad (3)$$

Subtracting (2) from (3),

$$\frac{8}{y} = 1$$

Whence, $y = 8$, the circumference of the fore-wheel.

Substituting in (1),

$$3 - \frac{24}{x} = 1$$

Whence, $x = 12$, the circumference of the hind-wheel.

61. Let $x =$ the number of days A takes to do the work,

and $y =$ the number B takes.

Then in one day A does $\dfrac{1}{x}$ and B $\dfrac{1}{y}$.

Also $\dfrac{1}{2x} =$ the part A does in one day if he works one-half as fast, and

$\dfrac{2}{y} =$ the part B does in one day if he works twice as fast.

By the conditions,

$$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{24} & (1) \\ \dfrac{1}{2x}+\dfrac{2}{y}=\dfrac{11}{48} & (2) \end{cases}$$

Multiplying (2) by 2,

$$\frac{1}{x}+\frac{4}{y}=\frac{11}{24} \qquad (3)$$

Subtracting (1) from (3),

$$\frac{3}{y}=\frac{1}{4}$$

Whence, $y = 12$, the number of days B takes to do the work.

Substituting in (1),

$$\frac{1}{x}+\frac{1}{12}=\frac{5}{24}$$

Whence, $x = 8$, the number of days A takes to do the work.

62. Let $x =$ the no. of yards A runs in a second,
and $y =$ the no. B runs.

The first heat, A runs 300 yards in 2 seconds less time than B runs 260 yards; the second heat, A runs 264 yards in 16 seconds less time than B runs 300 yards.

Then by the conditions,

$$\begin{cases} \dfrac{300}{x}+2=\dfrac{260}{y} & (1) \\ \dfrac{264}{x}+16=\dfrac{300}{y} & (2) \end{cases}$$

Multiplying (1) by $\frac{11}{2}$, and (2) by $\frac{25}{4}$,

$$\frac{1650}{x}-\frac{1430}{y}=-\ 11$$

$$\frac{1650}{x}-\frac{1875}{y}=-100$$

Subtracting, $\dfrac{445}{y}=89$

Whence, $y =\ 5$, the no. of yards B runs in a second.

Substituting in (1),

$$\frac{300}{x}=50$$

Whence, $x =\ 6$, the no. of yards A runs in a second.

CHAPTER XVII.

Art. 193. — Page 159.

3. $(-ab^2c^3)^4 = a^4b^8c^{12}$.
4. $(-5a^2b)^3 = -125a^6b^3$.
5. $(x^ny)^m = x^{mn}y^m$.
6. $(2mn^2x^3)^6 = 64m^6n^{12}x^{18}$.
7. $(-b^2c^3)^5 = -b^{10}c^{15}$.
8. $(a^2b^2c^m)^n = a^{2n}b^{2n}c^{mn}$.
9. $(-5m^3n)^4 = 625m^{12}n^4$.
10. $(4a^2b^3c^4)^3 = 64a^6b^9c^{12}$.
11. $(3a^3b^4c)^6 = 729a^{18}b^{24}c^6$.
12. $(-6x^ry^{7s})^3 = -216x^{3r}y^{21s}$.
13. $(4a^mb^{2n})^5 = 1024a^{5m}b^{10n}$.
14. $(-7x^6y^3z^{15})^3 = -343x^{18}y^9z^{45}$.
15. $\left(\frac{ac^2}{b^3}\right)^4 = \frac{a^4c^8}{b^{12}}$.
16. $\left(\frac{3a^2b^3}{4xy^4}\right)^3 = \frac{27a^6b^9}{64x^3y^{12}}$.
17. $\left(-\frac{4ax^2}{5b}\right)^2 = \frac{16a^2x^4}{25b^2}$.
18. $\left(\frac{2}{3}a^3x^2\right)^6 = \frac{64}{729}a^{18}x^{12}$.
19. $\left(-\frac{7xy^2}{3n}\right)^3 = -\frac{343x^3y^6}{27n^3}$.
20. $\left(-\frac{bcx^n}{4a^2}\right)^5 = -\frac{b^5c^5x^{5n}}{1024a^{10}}$.

Art. 194. — Page 160.

2. $(a-b+c)^2 = a^2+b^2+c^2-2ab+2ac-2bc$.
3. $(a+b-c)^2 = a^2+b^2+c^2+2ab-2ac-2bc$.
4. $(2x^2+x+1)^2 = 4x^4+x^2+1+4x^3+4x^2+2x$.
$= 4x^4+4x^3+5x^2+2x+1$.
5. $(x^2-3x+1)^2 = x^4+9x^2+1-6x^3+2x^2-6x$
$= x^4-6x^3+11x^2-6x+1$.
6. $(x^2+4x-2)^2 = x^4+16x^2+4+8x^3-4x^2-16x$
$= x^4+8x^3+12x^2-16x+4$.
7. $(2x^2-x-3)^2 = 4x^4+x^2+9-4x^3-12x^2+6x$
$= 4x^4-4x^3-11x^2+6x+9$.
8. $(3a^2-5a+4)^2 = 9a^4+25a^2+16-30a^3+24a^2-40a$
$= 9a^4-30a^3+49a^2-40a+16$.
9. $(2x^2+5x-7)^2 = 4x^4+25x^2+49+20x^3-28x^2-70x$
$= 4x^4+20x^3-3x^2-70x+49$.

10. $(x+2y-3z)^2 = x^2+4y^2+9z^2+4xy-6xz-12yz.$

11. $(x^3-2x+5)^2 = x^6+4x^2+25-4x^4+10x^3-20x$
$= x^6-4x^4+10x^3+4x^2-20x+25.$

12. $(2x^3+3x^2+1)^2 = 4x^6+9x^4+1+12x^5+4x^3+6x^2$
$= 4x^6+12x^5+9x^4+4x^3+6x^2+1.$

13. $(3a^2-2ab-5b^2)^2 = 9a^4+4a^2b^2+25b^4-12a^3b-30a^2b^2+20ab^3$
$= 9a^4-12a^3b-26a^2b^2+20ab^3+25b^4.$

14. $(4m^2+mn^2-3n^4)^2 = 16m^4+m^2n^4+9n^8+8m^3n^2-24m^2n^4-6mn^6$
$= 16m^4+8m^3n^2-23m^2n^4-6mn^6+9n^8.$

15. $(a-b-c+d)^2$
$= a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd.$

16. $(a-b+c-d)^2$
$= a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd.$

17. $(1+x+x^2+x^3)^2$
$= 1+x^2+x^4+x^6+2x+2x^2+2x^3+2x^3+2x^4+2x^5$
$= 1+2x+3x^2+4x^3+3x^4+2x^5+x^6.$

18. $(3x^3-2x^2-x+4)^2$
$= 9x^6+4x^4+x^2+16-12x^5-6x^4+24x^3+4x^3-16x^2-8x$
$= 9x^6-12x^5-2x^4+28x^3-15x^2-8x+16.$

19. $(x^3-4x^2-2x-3)^2$
$= x^6+16x^4+4x^2+9-8x^5-4x^4-6x^3+16x^3+24x^2+12x$
$= x^6-8x^5+12x^4+10x^3+28x^2+12x+9.$

Art. 195. — Page 162.

3. $(x+3)^3 = x^3+3x^2\cdot 3+3x\cdot 3^2+3^3 = x^3+9x^2+27x+27.$

4. $(2x-1)^3 = (2x)^3-3(2x)^2+3(2x)-1 = 8x^3-12x^2+6x-1.$

5. $(ab-cd)^3 = a^3b^3-3a^2b^2cd+3abc^2d^2-c^3d^3.$

6. $(a+4b)^3 = a^3+3a^2(4b)+3a(4b)^2+(4b)^3$
$= a^3+12a^2b+48ab^2+64b^3.$

7. $(3m^2-1)^3 = (3m^2)^3-3(3m^2)^2+3(3m^2)-1$
$= 27m^6-27m^4+9m^2-1.$

8. $(x^2+4)^3 = (x^2)^3+3(x^2)^2\cdot 4+3x^2\cdot 4^2+4^3 = x^6+12x^4+48x^2+64.$

9. $(a+5b)^3 = a^3+3a^2(5b)+3a(5b)^2+(5b)^3$
$= a^3+15a^2b+75ab^2+125b^3.$

10. $(2x-5y)^3 = (2x)^3-3(2x)^2(5y)+3(2x)(5y)^2-(5y)^3$
$= 8x^3-60x^2y+150xy^2-125y^3.$

11. $(2x^3-3x)^3 = (2x^3)^3-3(2x^3)^2(3x)+3(2x^3)(3x)^2-(3x)^3$
$= 8x^9-36x^7+54x^5-27x^3.$

12. $(6x^2 + xy)^3 = (6x^2)^3 + 3(6x^2)^2(xy) + 3(6x^2)(xy)^2 + (xy)^3$
$= 216x^6 + 108x^5y + 18x^4y^2 + x^3y^3.$

13. $(3m + 5n)^3 = (3m)^3 + 3(3m)^2(5n) + 3(3m)(5n)^2 + (5n)^3$
$= 27m^3 + 135m^2n + 225mn^2 + 125n^3.$

14. $(3xy - 4a^2)^3 = (3xy)^3 - 3(3xy)^2(4a^2) + 3(3xy)(4a^2)^2 - (4a^2)^3$
$= 27x^3y^3 - 108a^2x^2y^2 + 144a^4xy - 64a^6.$

16. $(x^2 - x - 1)^3 = [(x^2 - x) - 1]^3$
$= (x^2 - x)^3 - 3(x^2 - x)^2 + 3(x^2 - x) - 1$
$= x^6 - 3x^5 + 3x^4 - x^3 - 3(x^4 - 2x^3 + x^2) + 3(x^2 - x) - 1$
$= x^6 - 3x^5 + 5x^3 - 3x - 1.$

17. $(a - b + 1)^3 = [(a - b) + 1]^3 = (a - b)^3 + 3(a - b)^2 + 3(a - b) + 1$
$= a^3 - 3a^2b + 3ab^2 - b^3 + 3(a^2 - 2ab + b^2) + 3(a - b) + 1$
$= a^3 - 3a^2b + 3a^2 + 3ab^2 - 6ab + 3a - b^3 + 3b^2 - 3b + 1.$

18. $(a + b - c)^3 = [(a + b) - c]^3$
$= (a + b)^3 - 3(a + b)^2c + 3(a + b)c^2 - c^3$
$= a^3 + 3a^2b + 3ab^2 + b^3 - 3c(a^2 + 2ab + b^2) + 3c^2(a + b) - c^3$
$= a^3 + 3a^2b - 3a^2c + 3ab^2 - 6abc + 3ac^2 + b^3 - 3b^2c + 3bc^2 - c^3.$

19. $(x^2 - 2x + 2)^3 = [(x^2 - 2x) + 2]^3$
$= (x^2 - 2x)^3 + 3(x^2 - 2x)^2 \cdot 2 + 3(x^2 - 2x) \cdot 2^2 + 2^3$
$= x^6 - 6x^5 + 12x^4 - 8x^3 + 6(x^4 - 4x^3 + 4x^2) + 12(x^2 - 2x) + 8$
$= x^6 - 6x^5 + 18x^4 - 32x^3 + 36x^2 - 24x + 8.$

20. $(x^2 + 3x + 1)^3 = [(x^2 + 3x) + 1]^3$
$= (x^2 + 3x)^3 + 3(x^2 + 3x)^2 + 3(x^2 + 3x) + 1$
$= x^6 + 9x^5 + 27x^4 + 27x^3 + 3(x^4 + 6x^3 + 9x^2) + 3(x^2 + 3x) + 1$
$= x^6 + 9x^5 + 30x^4 + 45x^3 + 30x^2 + 9x + 1.$

21. $(2x^2 - 3x - 1)^3 = [(2x^2 - 3x) - 1]^3$
$= (2x^2 - 3x)^3 - 3(2x^2 - 3x)^2 + 3(2x^2 - 3x) - 1$
$= 8x^6 - 36x^5 + 54x^4 - 27x^3 - 3(4x^4 - 12x^3 + 9x^2) + 3(2x^2 - 3x) - 1$
$= 8x^6 - 36x^5 + 42x^4 + 9x^3 - 21x^2 - 9x - 1.$

Art. 196.—Page 164.

3. $(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$

4. $(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$

5. $(a - b)^7 = a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7.$

6. $(x - 1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1.$

7. $(1 - x)^4 = 1^4 - 4 \cdot 1^3 \cdot x + 6 \cdot 1^2 \cdot x^2 - 4 \cdot 1 \cdot x^3 + x^4$
$= 1 - 4x + 6x^2 - 4x^3 + x^4.$

8. $(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.$

9. $(m-n)^6 = m^6 - 6m^5n + 15m^4n^2 - 20m^3n^3 + 15m^2n^4 - 6mn^5 + n^6.$

10. $(2+x)^4 = 2^4 + 4\cdot 2^3\cdot x + 6\cdot 2^2\cdot x^2 + 4\cdot 2\cdot x^3 + x^4$
$= 16 + 32x + 24x^2 + 8x^3 + x^4.$

11. $(x-4)^4 = x^4 - 4x^3\cdot 4 + 6x^2\cdot 4^2 - 4x\cdot 4^3 + 4^4$
$= x^4 - 16x^3 + 96x^2 - 256x + 256.$

12. $(a-3)^5 = a^5 - 5a^4\cdot 3 + 10a^3\cdot 3^2 - 10a^2\cdot 3^3 + 5a\cdot 3^4 - 3^5$
$= a^5 - 15a^4 + 90a^3 - 270a^2 + 405a - 243.$

13. $(a+2)^5 = a^5 + 5a^4\cdot 2 + 10a^3\cdot 2^2 + 10a^2\cdot 2^3 + 5a\cdot 2^4 + 2^5$
$= a^5 + 10a^4 + 40a^3 + 80a^2 + 80a + 32.$

14. $(x-2)^6 = x^6 - 6x^5\cdot 2 + 15x^4\cdot 2^2 - 20x^3\cdot 2^3 + 15x^2\cdot 2^4 - 6x\cdot 2^5 + 2^6$
$= x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64.$

16. $(a-3x)^5 = a^5 - 5a^4(3x) + 10a^3(3x)^2 - 10a^2(3x)^3 + 5a(3x)^4 - (3x)^5$
$= a^5 - 15a^4x + 90a^3x^2 - 270a^2x^3 + 405ax^4 - 243x^5.$

17. $(3+2b)^4 = 3^4 + 4\cdot 3^3\cdot(2b) + 6\cdot 3^2\cdot(2b)^2 + 4\cdot 3\cdot(2b)^3 + (2b)^4$
$= 81 + 216b + 216b^2 + 96b^3 + 16b^4.$

18. $(a^2+bc)^7 = (a^2)^7 + 7(a^2)^6bc + 21(a^2)^5b^2c^2 + 35(a^2)^4b^3c^3$
$+ 35(a^2)^3b^4c^4 + 21(a^2)^2b^5c^5 + 7a^2b^6c^6 + b^7c^7$
$= a^{14} + 7a^{12}bc + 21a^{10}b^2c^2 + 35a^8b^3c^3 + 35a^6b^4c^4$
$+ 21a^4b^5c^5 + 7a^2b^6c^6 + b^7c^7.$

19. $(x^3-4)^4 = (x^3)^4 - 4(x^3)^3\cdot 4 + 6(x^3)^2\cdot 4^2 - 4(x^3)\cdot 4^3 + 4^4$
$= x^{12} - 16x^9 + 96x^6 - 256x^3 + 256.$

20. $(2a^2+b)^6 = (2a^2)^6 + 6(2a^2)^5b + 15(2a^2)^4b^2 + 20(2a^2)^3b^3$
$+ 15(2a^2)^2b^4 + 6(2a^2)b^5 + b^6$
$= 64a^{12} + 192a^{10}b + 240a^8b^2 + 160a^6b^3 + 60a^4b^4 + 12a^2b^5 + b^6.$

21. $(2m^3-3n^2)^4 = (2m^3)^4 - 4(2m^3)^3(3n^2) + 6(2m^3)^2(3n^2)^2$
$- 4(2m^3)(3n^2)^3 + (3n^2)^4$
$= 16m^{12} - 96m^9n^2 + 216m^6n^4 - 216m^3n^6 + 81n^8.$

CHAPTER XVIII.

Art. 202.—Pages 166, 167.

3. $\sqrt[3]{-125x^3y^6} = -5xy^2.$

4. $\sqrt{49a^4b^2c^{12}} = \pm 7a^2bc^6.$

5. $\sqrt[5]{m^{15}n^5p^{10}} = m^3np^2.$

6. $\sqrt[4]{16a^4b^8} = \pm 2ab^2.$

7. $\sqrt[3]{-8a^3b^6x^9} = -2ab^2x^3.$

8. $\sqrt{121a^{12}c^2} = \pm 11a^6c.$

9. $\sqrt[n]{a^{mn}b^{np}} = a^mb^p.$

10. $\sqrt{81a^{2n}x^{2m+2}} = \pm 9a^nx^{m+1}.$

11. $\sqrt[4]{81m^{16}n^{20}} = \pm 3m^4n^5.$

12. $\sqrt[5]{-243c^{5n}d^{10m}} = -3c^nd^{2m}.$

13. $\sqrt[6]{64a^{18}b^{24}c^6} = \pm 2a^3b^4c.$

14. $\sqrt[3]{x^{3n+3}y^{9m-6}} = x^{n+1}y^{3m-2}.$

15. $\sqrt{\dfrac{9x^2y^4}{16m^6}} = \pm\dfrac{3xy^2}{4m^3}.$

16. $\sqrt[3]{\dfrac{8a^3b^9}{27c^6}} = \dfrac{2ab^3}{3c^2}.$

17. $\sqrt[5]{-\dfrac{32x^{15}}{y^{10}}} = -\dfrac{2x^3}{y^2}.$

18. $\sqrt[4]{\dfrac{a^4}{81b^8c^4}} = \pm\dfrac{a}{3b^2c}.$

19. $\sqrt[3]{-\dfrac{64m^3n^6}{125}} = -\dfrac{4mn^2}{5}.$

20. $\sqrt[5]{\dfrac{a^{5m}}{243x^{10}}} = \dfrac{a^m}{3x^2}.$

Art. 204.—Page 169.

3.

$$
\begin{array}{l|l|l}
 & a^4 - 4a^3 + 6a^2 - 4a + 1 & a^2 - 2a + 1 \\
 & a^4 & \\
\hline
2a^2 - 2a & -4a^3 & \\
 & -4a^3 + 4a^2 & \\
\hline
2a^2 - 4a + 1 & 2a^2 - 4a + 1 & \\
 & 2a^2 - 4a + 1 & \\
\hline
\end{array}
$$

4.

$$
\begin{array}{l|l|l}
 & 4x^4 - 4x^3 - 3x^2 + 2x + 1 & 2x^2 - x - 1 \\
 & 4x^4 & \\
\hline
4x^2 - x & -4x^3 & \\
 & -4x^3 + x^2 & \\
\hline
4x^2 - 2x - 1 & -4x^2 + 2x + 1 & \\
 & -4x^2 + 2x + 1 & \\
\hline
\end{array}
$$

5.

$$\begin{array}{l|l}
9-12x+10x^2-4x^3+x^4 & 3-2x+x^2 \\
9 & \\ \hline
\end{array}$$

$$\begin{array}{l|l}
6-2x & -12x \\
 & -12x+4x^2 \\ \hline
6-4x+x^2 & 6x^2-4x^3+x^4 \\
 & 6x^2-4x^3+x^4 \\ \hline
\end{array}$$

6.

$$\begin{array}{l|l}
25+30x+19x^2+6x^3+x^4 & 5+3x+x^2 \\
25 & \\ \hline
\end{array}$$

$$\begin{array}{l|l}
10+3x & 30x \\
 & 30x+9x^2 \\ \hline
10+6x+x^2 & 10x^2+6x^3+x^4 \\
 & 10x^2+6x^3+x^4 \\ \hline
\end{array}$$

7.

$$\begin{array}{l|l}
9x^4-24x^3-14x^2+40x+25 & 3x^2-4x-5 \\
9x^4 & \\ \hline
\end{array}$$

$$\begin{array}{l|l}
6x^2-4x & -24x^3 \\
 & -24x^3+16x^2 \\ \hline
6x^2-8x-5 & -30x^2+40x+25 \\
 & -30x^2+40x+25 \\ \hline
\end{array}$$

8.

$$\begin{array}{l|l}
m^2+2m-1-\frac{2}{m}+\frac{1}{m^2} & m+1-\frac{1}{m} \\
m^2 & \\ \hline
\end{array}$$

$$\begin{array}{l|l}
2m+1 & 2m \\
 & 2m+1 \\ \hline
2m+2-\frac{1}{m} & -2-\frac{2}{m}+\frac{1}{m^2} \\
 & -2-\frac{2}{m}+\frac{1}{m^2} \\ \hline
\end{array}$$

9.

$$\begin{array}{l|l}
4a^4-20a^3b+57a^2b^2-80ab^3+64b^4 & 2a^2-5ab+8b^2 \\
4a^4 & \\ \hline
\end{array}$$

$$\begin{array}{l|l}
4a^2-5ab & -20a^3b \\
 & -20a^3b+25a^2b^2 \\ \hline
4a^2-10ab+8b^2 & 32a^2b^2-80ab^3+64b^4 \\
 & 32a^2b^2-80ab^3+64b^4 \\ \hline
\end{array}$$

10.

$$
\begin{array}{l|l|l}
 & 1-14x+45x^2+28x^3+4x^4 & 1-7x-2x^2 \\
 & 1 & \\
\hline
2-7x & -14x & \\
 & -14x+49x^2 & \\
\hline
2-14x-2x^2 & -4x^2+28x^3+4x^4 & \\
 & -4x^2+28x^3+4x^4 & \\
\hline
\end{array}
$$

11.

$$
\begin{array}{l|l|l}
 & a^2-2ab-2ac+b^2+2bc+c^2 & a-b-c \\
 & a^2 & \\
\hline
2a-b & -2ab & \\
 & -2ab \quad +b^2 & \\
\hline
2a-2b-c & -2ac \quad +2bc+c^2 & \\
 & -2ac \quad +2bc+c^2 & \\
\hline
\end{array}
$$

12.

$$
\begin{array}{l|l|l}
 & x^2-4xy+6xz+4y^2-12yz+9z^2 & x-2y+3z \\
 & x^2 & \\
\hline
2x-2y & -4xy & \\
 & -4xy \quad +4y^2 & \\
\hline
2x-4y+3z & 6xz \quad -12yz+9z^2 & \\
 & 6xz \quad -12yz+9z^2 & \\
\hline
\end{array}
$$

13.

$$
\begin{array}{l|l|l}
 & 9x^6+30x^5+25x^4-42x^3-70x^2+49 & 3x^3+5x^2-7 \\
 & 9x^6 & \\
\hline
6x^3+5x^2 & 30x^5 & \\
 & 30x^5+25x^4 & \\
\hline
6x^3+10x^2-7 & -42x^3-70x^2+49 & \\
 & -42x^3-70x^2+49 & \\
\hline
\end{array}
$$

14.

$$
\begin{array}{l|l|l}
 & 16c^6-40c^4-24c^3+25c^2+30c+9 & 4c^3-5c-3 \\
 & 16c^6 & \\
\hline
8c^3-5c & -40c^4 & \\
 & -40c^4 \quad +25c^2 & \\
\hline
8c^3-10c-3 & -24c^3 \quad +30c+9 & \\
 & -24c^3 \quad +30c+9 & \\
\hline
\end{array}
$$

15.

$$
\begin{array}{l|l|l}
 & a^6 - 4a^5 + 14a^4 - 14a^3 + 13a^2 + 30a + 9 & a^3 - 2a^2 + 5a + 3 \\
 & a^6 & \\
\hline
2a^3 - 2a^2 & -4a^5 & \\
 & -4a^5 + 4a^4 & \\
\hline
2a^3 - 4a^2 + 5a & 10a^4 & \\
 & 10a^4 - 20a^3 + 25a^2 & \\
\hline
2a^3 - 4a^2 + 10a + 3 & 6a^3 - 12a^2 + 30a + 9 & \\
 & 6a^3 - 12a^2 + 30a + 9 & \\
\hline
\end{array}
$$

16.

$$
\begin{array}{l|l|l}
 & 4x^6 - 4x^5y - 3x^4y^2 - 6x^3y^3 + 5x^2y^4 + 4xy^5 + 4y^6 & 2x^3 - x^2y - xy^2 - 2y^3 \\
 & 4x^6 & \\
\hline
4x^3 - x^2y & -4x^5y & \\
 & -4x^5y + x^4y^2 & \\
\hline
4x^3 - 2x^2y - xy^2 & -4x^4y^2 & \\
 & -4x^4y^2 + 2x^3y^3 + x^2y^4 & \\
\hline
4x^3 - 2x^2y - 2xy^2 - 2y^3 & -8x^3y^3 + 4x^2y^4 + 4xy^5 + 4y^6 & \\
 & -8x^3y^3 + 4x^2y^4 + 4xy^5 + 4y^6 & \\
\hline
\end{array}
$$

17.

$$
\begin{array}{l|l|l}
 & 4x^6 - 12x^5 + 25x^4 - 44x^3 + 46x^2 - 40x + 25 & 2x^3 - 3x^2 + 4x - 5 \\
 & 4x^6 & \\
\hline
4x^3 - 3x^2 & -12x^5 & \\
 & -12x^5 + 9x^4 & \\
\hline
4x^3 - 6x^2 + 4x & 16x^4 & \\
 & 16x^4 - 24x^3 + 16x^2 & \\
\hline
4x^3 - 6x^2 + 8x - 5 & -20x^3 + 30x^2 - 40x + 25 & \\
 & -20x^3 + 30x^2 - 40x + 25 & \\
\hline
\end{array}
$$

18.

$$
\begin{array}{l|l|l}
 & \frac{a^4}{9} - \frac{2a^3b}{3} + \frac{4a^2b^2}{3} - ab^3 + \frac{b^4}{4} & \frac{a^2}{3} - ab + \frac{b^2}{2} \\
 & \frac{a^4}{9} & \\
\hline
\frac{2a^2}{3} - ab & -\frac{2a^3b}{3} & \\
 & -\frac{2a^3b}{3} + a^2b^2 & \\
\hline
\frac{2a^2}{3} - 2ab + \frac{b^2}{2} & \frac{a^2b^2}{3} & \\
 & \frac{a^2b^2}{3} - ab^3 + \frac{b^4}{4} & \\
\hline
\end{array}
$$

19.

$$
\begin{array}{l|l}
 & 9x^6-12x^5y+10x^4y^2-16x^3y^3+9x^2y^4-4xy^5+4y^6 \;\big|\; 3x^3-2x^2y+xy^2-2y^3 \\
 & 9x^6 \\
\hline
6x^3-2x^2y & -12x^5y \\
 & -12x^5y+4x^4y^2 \\
\hline
6x^3-4x^2y+xy^2 & 6x^4y^2 \\
 & 6x^4y^2-4x^3y^3+x^2y^4 \\
\hline
6x^3-4x^2y+2xy^2-2y^3 & -12x^3y^3+8x^2y^4-4xy^5+4y^6 \\
 & -12x^3y^3+8x^2y^4-4xy^5+4y^6 \\
\hline
\end{array}
$$

20.

$$
\begin{array}{l|l}
 & 1+x \;\big|\; 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\cdots \\
 & 1 \\
\hline
2+\frac{x}{2} & x \\
 & x+\frac{x^2}{4} \\
\hline
2+x-\frac{x^2}{8} & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
\hline
2+x-\frac{x^2}{4}+\frac{x^3}{16} & \frac{x^3}{8}-\frac{x^4}{64} \\
 & \frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256} \\
\hline
 & -\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256}
\end{array}
$$

21.

$$
\begin{array}{l|l}
 & 1-2a \;\big|\; 1-a-\frac{a^2}{2}-\frac{a^3}{2}-\cdots \\
 & 1 \\
\hline
2-a & -2a \\
 & -2a+a^2 \\
\hline
2-2a-\frac{a^2}{2} & -a^2 \\
 & -a^2+a^3+\frac{a^4}{4} \\
\hline
2-2a-a^2-\frac{a^3}{2} & -a^3-\frac{a^4}{4} \\
 & -a^3+a^4+\frac{a^5}{2}+\frac{a^6}{4} \\
\hline
 & -\frac{5a^4}{4}-\frac{a^5}{2}-\frac{a^6}{4}
\end{array}
$$

22.

$$\begin{array}{l|l|l}
 & a^2-4ab+b^2 & a-2b-\frac{3b^2}{2a}-\frac{3b^3}{a^2}-\cdots \\
 & a^2 & \\
\hline
2a-2b & -4ab+b^2 & \\
 & -4ab+4b^2 & \\
\hline
2a-4b-\frac{3b^2}{2a} & -3b^2 & \\
 & -3b^2+\frac{6b^3}{a}+\frac{9b^4}{4a^2} & \\
\hline
2a-4b-\frac{3b^2}{a}-\frac{3b^3}{a^2} & -\frac{6b^3}{a}-\frac{9b^4}{4a^2} & \\
 & -\frac{6b^3}{a}+\frac{12b^4}{a^2}+\frac{9b^5}{a^3}+\frac{9b^6}{a^4} & \\
\hline
 & -\frac{57b^4}{4a^2}-\frac{9b^5}{a^3}-\frac{9b^6}{a^4} &
\end{array}$$

23.

$$\begin{array}{l|l|l}
 & 4x^2+2y & 2x+\frac{y}{2x}-\frac{y^2}{16x^3}+\frac{y^3}{64x^5}-\cdots \\
 & 4x^2 & \\
\hline
4x+\frac{y}{2x} & 2y & \\
 & 2y+\frac{y^2}{4x^2} & \\
\hline
4x+\frac{y}{x}-\frac{y^2}{16x^3} & -\frac{y^2}{4x^2} & \\
 & -\frac{y^2}{4x^2}-\frac{y^3}{16x^4}+\frac{y^4}{256x^6} & \\
\hline
4x+\frac{y}{x}-\frac{y^2}{8x^3}+\frac{y^3}{64x^5} & \frac{y^3}{16x^4}-\frac{y^4}{256x^6} & \\
 & \frac{y^3}{16x^4}+\frac{y^4}{64x^6}-\cdots & \\
\hline
 & -\frac{5y^4}{256x^6}+\cdots &
\end{array}$$

Art. 207. — Pages 172, 173.

2.

$$\begin{array}{r|l|l}
 & \dot{4}57\dot{9}6 & 214 \\
 & 4 & \\
\hline
41 & 57 & \\
 & 41 & \\
\hline
424 & 1696 & \\
 & 1696 &
\end{array}$$

3.

$$\begin{array}{r|l|l}
 & 2\dot{7}3\dot{5}2\dot{9} & 523 \\
 & 25 & \\
\hline
102 & 235 & \\
 & 204 & \\
\hline
1043 & 3129 & \\
 & 3129 &
\end{array}$$

4.

654481 | 809
64
1609 | 14481
 | 14481

5.

33.1776 | 5.76
25
107 | 8 17
 | 7 49
1146 | 6876
 | 6876

6.

.247009 | .497
16
89 | 870
 | 801
987 | 6909
 | 6909

7.

.081796 | .286
4
48 | 417
 | 384
566 | 3396
 | 3396

8.

.521284 | .722
49
142 | 312
 | 284
1442 | 2884
 | 2884

9.

1.170724 | 1.082
1
208 | 1707
 | 1664
2162 | 4324
 | 4324

10.

446.0544 | 21.12
4
41 | 46
 | 41
421 | 505
 | 421
4222 | 8444
 | 8444

11.

.00224486444 | .04738
16
87 | 644
 | 609
943 | 3586
 | 2829
9468 | 75744
 | 75744

12.

811440.64 | 900.8
81
18008 | 144064
 | 144064

13.

.68112009 | .8253
64
162 | 411
 | 324
1645 | 8720
 | 8225
16503 | 49509
 | 49509

15.

2. | 1.4142, approx.
1
24 | 1 00
 | 96
281 | 400
 | 281
2824 | 11900
 | 11296
2828 | 60400

16.

	$\dot{3}.$	1.7321, approx.
	1	
27	200	
	189	
343	1100	
	1029	
3462	7100	
	6924	
3464	17600	

17.

	$\dot{5}.$	2.2361, approx.
	4	
42	100	
	84	
443	1600	
	1329	
4466	27100	
	26796	
4472	30400	

18.

	$1\dot{1}.$	3.3166, approx.
	9	
63	200	
	189	
661	1100	
	661	
6626	43900	
	39756	
6632	414400	

19.

	$3\dot{1}.$	5.5678, approx.
	25	
105	600	
	525	
1106	7500	
	6636	
11127	86400	
	77889	
11134	851100	

20.

	$1\dot{7}.3\dot{0}$	4.1593, approx.
	16	
81	1 30	
	81	
825	4900	
	4125	
8309	77500	
	74781	
8318	271900	

21.

	$.7\dot{0}$	.83666, approx.
	64	
163	600	
	489	
1666	11100	
	9996	
16726	110400	
	100356	
16732	1004400	

22.

	$.0\dot{8}$	.28284, approx.
	4	
48	400	
	384	
562	1600	
	1124	
5648	47600	
	45184	
5656	241600	

23.

	$.1\dot{4}4\dot{0}$	.37947, approx.
	9	
67	540	
	469	
749	7100	
	6741	
7584	35900	
	30336	
7588	556400	

24.

$$\begin{array}{r|l}
 & .0\dot{0}1\dot{0} \quad | \; .031623, \text{ approx.} \\
 & 9 \\
\hline
61 & 100 \\
 & 61 \\
\hline
626 & 3900 \\
 & 3756 \\
\hline
6322 & 14400 \\
 & 12644 \\
\hline
6324 & 175600
\end{array}$$

25.

$$\begin{array}{r|l}
 & .0\dot{0}6\dot{2}5\dot{0} \quad | \; .079057, \text{ approx.} \\
 & 49 \\
\hline
149 & 1350 \\
 & 1341 \\
\hline
15805 & 90000 \\
 & 79025 \\
\hline
15810 & 1097500
\end{array}$$

26.

$$\begin{array}{r|l}
 & \dot{2}.0\dot{8}6\dot{2}7\dot{0} \quad | \; 1.4444, \text{ approx.} \\
 & 1 \\
\hline
24 & 1\,08 \\
 & 96 \\
\hline
284 & 1262 \\
 & 1136 \\
\hline
2884 & 12670 \\
 & 11536 \\
\hline
2888 & 113400
\end{array}$$

27.

$$\sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{2} = \frac{2.6457\cdots}{2} = 1.3229, \text{ approx.}$$

$$\begin{array}{r|l}
 & \dot{7}. \quad | \; 2.6457\cdots \\
 & 4 \\
\hline
46 & 300 \\
 & 276 \\
\hline
524 & 2400 \\
 & 2096 \\
\hline
5285 & 30400 \\
 & 26425 \\
\hline
5290 & 397500
\end{array}$$

28.

$$\sqrt{\frac{3}{16}} = \frac{\sqrt{3}}{4} = \frac{1.73205\cdots}{4} = .43301, \text{ approx.}$$

$$\begin{array}{r|l}
 & \dot{3}. \quad | \; 1.73205\cdots \\
 & 1 \\
\hline
27 & 200 \\
 & 189 \\
\hline
343 & 1100 \\
 & 1029 \\
\hline
3462 & 7100 \\
 & 6924 \\
\hline
34640 & 1760000
\end{array}$$

29.

$$\sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3} = \frac{3.1622\cdots}{3} = 1.0541, \text{ approx.}$$

$$\begin{array}{r|l}
 & 1\dot{0}. \quad | \; 3.1622\cdots \\
 & 9 \\
\hline
61 & 100 \\
 & 61 \\
\hline
626 & 3900 \\
 & 3756 \\
\hline
6322 & 14400 \\
 & 12644 \\
\hline
6324 & 175600
\end{array}$$

30.

$$\sqrt{\frac{1}{5}} = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5} = \frac{2.23606\cdots}{5}$$

$$= .44721, \text{ approx.}$$

$$\begin{array}{r|l}
 & \dot{5}. \quad | \; 2.23606\cdots \\
 & 4 \\
\hline
42 & 100 \\
 & 84 \\
\hline
443 & 1600 \\
 & 1329 \\
\hline
4466 & 27100 \\
 & 26796 \\
\hline
44720 & 3040000
\end{array}$$

31.

$$\sqrt{\frac{4}{3}}=\sqrt{\frac{12}{9}}=\frac{\sqrt{12}}{3}=\frac{3.4641\cdots}{3}$$
$= 1.1547$, approx.

$$\begin{array}{r|l}
 & 12. \quad \big|\ 3.4641\cdots \\
 & 9 \\ \hline
64 & 300 \\
 & 256 \\ \hline
686 & 4400 \\
 & 4116 \\ \hline
6924 & 28400 \\
 & 27696 \\ \hline
6928 & 70400
\end{array}$$

32.

$$\sqrt{\frac{5}{12}}=\sqrt{\frac{15}{36}}=\frac{\sqrt{15}}{6}=\frac{3.87298\cdots}{6}$$
$= .64550$, approx.

$$\begin{array}{r|l}
 & 15. \quad \big|\ 3.87298\cdots \\
 & 9 \\ \hline
68 & 600 \\
 & 544 \\ \hline
767 & 5600 \\
 & 5369 \\ \hline
7742 & 23100 \\
 & 15484 \\ \hline
77449 & 761600 \\
 & 697041 \\ \hline
77458 & 6455900
\end{array}$$

33.

$$\sqrt{\frac{11}{8}}=\sqrt{\frac{22}{16}}=\frac{\sqrt{22}}{4}=\frac{4.6904\cdots}{4}$$
$= 1.1726$, approx.

$$\begin{array}{r|l}
 & 22. \quad \big|\ 4.6904\cdots \\
 & 16 \\ \hline
86 & 600 \\
 & 516 \\ \hline
929 & 8400 \\
 & 8361 \\ \hline
9380 & 390000
\end{array}$$

34.

$$\sqrt{\frac{9}{10}}=\sqrt{\frac{90}{100}}=\frac{\sqrt{90}}{10}=\frac{9.4868\cdots}{10}$$
$= .94868$, approx.

$$\begin{array}{r|l}
 & 90. \quad \big|\ 9.4868\cdots \\
 & 81 \\ \hline
184 & 900 \\
 & 736 \\ \hline
1888 & 16400 \\
 & 15104 \\ \hline
18966 & 129600 \\
 & 113796 \\ \hline
18972 & 1580400
\end{array}$$

35.

$$\sqrt{\frac{7}{18}}=\sqrt{\frac{14}{36}}=\frac{\sqrt{14}}{6}=\frac{3.74165\cdots}{6}$$
$= .62361$, approx.

$$\begin{array}{r|l}
 & 14. \quad \big|\ 3.74165\cdots \\
 & 9 \\ \hline
67 & 500 \\
 & 469 \\ \hline
744 & 3100 \\
 & 2976 \\ \hline
7481 & 12400 \\
 & 7481 \\ \hline
74826 & 491900 \\
 & 448956 \\ \hline
74832 & 4294400
\end{array}$$

36.

$$\sqrt{\frac{13}{72}}=\sqrt{\frac{26}{144}}=\frac{\sqrt{26}}{12}=\frac{5.09901\cdots}{12}$$
$= .42492$, approx.

$$\begin{array}{r|l}
 & 26. \quad \big|\ 5.09901\cdots \\
 & 25 \\ \hline
1009 & 10000 \\
 & 9081 \\ \hline
10189 & 91900 \\
 & 91701 \\ \hline
101980 & 1990000
\end{array}$$

Art. 209. — Page 175.

3.

$$\begin{array}{r|l}
 & 1-6y+12y^2-8y^3 \quad | \; 1-2y \\
 & 1 \\
\hline
3-6y+4y^2 & -6y \\
 & -6y+12y^2-8y^3 \\
\hline
\end{array}$$

4.

$$\begin{array}{r|l}
 & 27x^6+27x^4+9x^2+1 \quad | \; 3x^2+1 \\
 & 27x^6 \\
\hline
27x^4+9x^2+1 & 27x^4 \\
 & 27x^4+9x^2+1 \\
\hline
\end{array}$$

5.

$$\begin{array}{r|l}
 & 8x^3+36x^2y+54xy^2+27y^3 \quad | \; 2x+3y \\
 & 8x^3 \\
\hline
12x^2+18xy+9y^2 & 36x^2y \\
 & 36x^2y+54xy^2+27y^3 \\
\hline
\end{array}$$

6.

$$\begin{array}{r|l}
 & 64a^3-144a^2xy+108ax^2y^2-27x^3y^3 \quad | \; 4a-3xy \\
 & 64a^3 \\
\hline
48a^2-36axy+9x^2y^2 & -144a^2xy \\
 & -144a^2xy+108ax^2y^2-27x^3y^3 \\
\hline
\end{array}$$

7.

$$\begin{array}{r|l}
 & x^6+6x^5 \qquad -40x^3+96x-64 \quad | \; x^2+2x-4 \\
 & x^6 \\
\hline
3x^4+6x^3+4x^2 & 6x^5 \\
 & 6x^5+12x^4+8x^3 \\
\hline
3x^4+12x^3+12x^2 & -12x^4-48x^3+96x-64 \\
-12x^2-24x+16 & \\
\hline
3x^4+12x^3 \qquad -24x+16 & -12x^4-48x^3+96x-64 \\
\hline
\end{array}$$

8.

$$\begin{array}{r|l}
 & y^6-3y^5 \qquad +5y^3-3y-1 \quad | \; y^2-y-1 \\
 & y^6 \\
\hline
3y^4-3y^3+y^2 & -3y^5 \\
 & -3y^5+3y^4-y^3 \\
\hline
3y^4-6y^3+3y^2 & -3y^4+6y^3-3y-1 \\
-3y^2+3y+1 & \\
\hline
3y^4-6y^3 \qquad +3y+1 & -3y^4+6y^3-3y-1 \\
\hline
\end{array}$$

9.

$$\begin{array}{r|l}
x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 & x^2 - 2x + 1 \\
x^6 & \\
\hline
3x^4 - 6x^3 + 4x^2 & -6x^5 \\
 & -6x^5 + 12x^4 - 8x^3 \\
\hline
3x^4 - 12x^3 + 12x^2 & 3x^4 - 12x^3 \\
3x^2 - 6x + 1 & \\
\hline
3x^4 - 12x^3 + 15x^2 - 6x + 1 & 3x^4 - 12x^3 + 15x^2 - 6x +
\end{array}$$

10.

$$\begin{array}{r|l}
8x^6 - 36x^5 + 42x^4 + 9x^3 - 21x^2 - 9x - 1 & 2x^2 - 3x - 1 \\
8x^6 & \\
\hline
12x^4 - 18x^3 + 9x^2 & -36x^5 \\
 & -36x^5 + 54x^4 - 27x^3 \\
\hline
12x^4 - 36x^3 + 27x^2 & -12x^4 + 36x^3 \\
-6x^2 + 9x + 1 & \\
\hline
12x^4 - 36x^3 + 21x^2 + 9x + 1 & -12x^4 + 36x^3 - 21x^2 - 9x - 1
\end{array}$$

11.

$$\begin{array}{r|l}
8a^6 - 12a^5 - 54a^4 + 59a^3 + 135a^2 - 75a - 125 & 2a^2 - a - 5 \\
8a^6 & \\
\hline
12a^4 - 6a^3 + a^2 & -12a^5 \\
 & -12a^5 + 6a^4 - a^3 \\
\hline
12a^4 - 12a^3 + 3a^2 & -60a^4 + 60a^3 \\
-30a^2 + 15a + 25 & \\
\hline
12a^4 - 12a^3 - 27a^2 + 15a + 25 & -60a^4 + 60a^3 + 135a^2 - 75a - 125
\end{array}$$

12.

$$\begin{array}{r|l}
8 - 12x + 30x^2 - 25x^3 + 30x^4 - 12x^5 + 8x^6 & 2 - x + 2x^2 \\
8 & \\
\hline
12 - 6x + x^2 & -12x \\
 & -12x + 6x^2 - x^3 \\
\hline
12 - 12x + 3x^2 & 24x^2 - 24x^3 \\
12x^2 - 6x^3 + 4x^4 & \\
\hline
12 - 12x + 15x^2 - 6x^3 + 4x^4 & 24x^2 - 24x^3 + 30x^4 - 12x^5 + 8x^6
\end{array}$$

13.

$$\begin{array}{r|l}
x^6 + 3x^5y - 3x^4y^2 - 11x^3y^3 + 6x^2y^4 + 12xy^5 - 8y^6 & x^2 + xy - 2y^2 \\
x^6 & \\
\hline
3x^4 + 3x^3y + x^2y^2 & 3x^5y \\
 & 3x^5y + 3x^4y^2 + x^3y^3 \\
\hline
3x^4 + 6x^3y + 3x^2y^2 & -6x^4y^2 - 12x^3y^3 \\
-6x^2y^2 - 6xy^3 + 4y^4 & \\
\hline
3x^4 + 6x^3y - 3x^2y^2 - 6xy^3 + 4y^4 & -6x^4y^2 - 12x^3y^3 + 6x^2y^4 + 12xy^5 - 8y^6
\end{array}$$

14.

$$\begin{array}{l|l}
\multicolumn{1}{r|}{27a^6-54a^5b+9a^4b^2+28a^3b^3-3a^2b^4-6ab^5-b^6} & 3a^2-2ab-b^2 \\
\multicolumn{1}{r|}{27a^6} & \\
\hline
\end{array}$$

$$\begin{array}{l|l}
27a^4-18a^3b+4a^2b^2 & -54a^5b \\
 & -54a^5b+36a^4b^2-8a^3b^3 \\
\hline
27a^4-36a^3b+12a^2b^2 & -27a^4b^2+36a^3b^3 \\
\qquad -9a^2b^2+6ab^3+b^4 & \\
\hline
27a^4-36a^3b+3a^2b^2+6ab^3+b^4 & -27a^4b^2+36a^3b^3-3a^2b^4-6ab^5-b^6 \\
\hline
\end{array}$$

Art. 213.—Pages 178, 179.

2.

$$\begin{array}{r|l}
2\dot{9}79\dot{1} & 31 \\
27 & \\
\hline
\end{array}$$

$$\begin{array}{r|r}
2700 & 2791 \\
90 & \\
1 & \\
\hline
2791 & 2791 \\
\hline
\end{array}$$

3.

$$\begin{array}{r|l}
9\dot{7}.33\dot{6} & 4.6 \\
64 & \\
\hline
\end{array}$$

$$\begin{array}{r|r}
4800 & 33\,336 \\
720 & \\
36 & \\
\hline
5556 & 33\,336 \\
\hline
\end{array}$$

4.

$$\begin{array}{r|l}
.68\dot{1}47\dot{2} & .88 \\
512 & \\
\hline
\end{array}$$

$$\begin{array}{r|r}
19200 & 169472 \\
1920 & \\
64 & \\
\hline
21184 & 169472 \\
\hline
\end{array}$$

5.

$$\begin{array}{r|l}
\dot{1}86\dot{0}86\dot{7} & 123 \\
1 & \\
\hline
\end{array}$$

$$\begin{array}{r|r}
300 & 860 \\
60 & \\
4 & \\
\hline
364 & 728 \\
\cline{2-2}
60 & 132867 \\
8 & \\
\hline
43200 & \\
1080 & \\
9 & \\
\hline
44289 & 132867 \\
\hline
\end{array}$$

6.

$$\begin{array}{r|l}
\dot{1}.48\dot{1}54\dot{4} & 1.14 \\
1 & \\
\hline
\end{array}$$

$$\begin{array}{r|r}
300 & 481 \\
30 & \\
1 & \\
\hline
331 & 331 \\
\cline{2-2}
30 & 150544 \\
2 & \\
\hline
36300 & \\
1320 & \\
16 & \\
\hline
37636 & 150544 \\
\hline
\end{array}$$

7.

	.000941192	.098
	729	
24300	212192	
2160		
64		
26524	212192	

8.

	8.242408	2.02
	8	
120000	242408	
1200		
4		
121204	242408	

9.

	51478848	372
	27	
2700	24478	
630		
49		
3379	23653	
630	825848	
98		
410700		
2220		
4		
412924	825848	

10.

	10077.696	21.6
	8	
1200	2077	
60		
1		
1261	1261	
60	816696	
2		
132300		
3780		
36		
136116	816696	

11.

	.517781627	.803
	512	
1920000	5781627	
7200		
9		
1927209	5781627	

12.

	116.930169	4.89
	64	
4800	52 930	
960		
64		
5824	46 592	
960	6 338169	
128		
691200		
12960		
81		
704241	6 338169	

13.

	.031855013	.317
	27	
2700	4855	
90		
1		
2791	2791	
90	2064013	
2		
288300		
6510		
49		
294859	2064013	

14.

```
                 .724150792 | .898
                 512
2397724 ... 
```

15.

```
          1039509.197 | 101.8
          1
30000   | 39509
  300   |
    1   | 30301
30301   |  9208197
  300   |
    2   |
3060300 |
   9090 |
      9 |
3069399 |  9208197
```

16.

```
        .000152273304 | .0534
            125
7500    | 27273
 450    |
   9    |
7959    | 23877
 450    |  3396304
  18    |
842700  |
  6360  |
    16  |
849076  |  3396304
```

17.

```
                 2.  | 1.2599...
                 1   | = 1.260, approx.
300        | 1000
 60        |
  4        |
364        |   728
 60        |   272000
  8        |
43200      |
 1800      |
   25      |
45025      |   225125
 1800      |    46875000
   50      |
4687500    |
  33750    |
     81    |
4721331    |    42491979
  33750    |     4383021000
    162    |
475524300  |
```

18.

```
                6.  | 1.817, approx.
                1
300      | 5000
240      |
 64      |
604      | 4832
240      |  168000
128      |
97200    |
  540    |
    1    |
97741    |  97741
  540    |   70259000
    2    |
9828300  |
```

19.

	$\dot{7}.20\dot{0}$	1.931, approx.
	1	
300	6200	
270		
81		
651	5859	
270	341000	
162		
108300		
1710		
9		
110019	330057	
1710	10943000	
18		
11174700		

	$\dot{3}.$	1.4422 ...
	1	
300	2000	
120		
16		
436	1744	
120	256000	
32		
58800		
1680		
16		
60496	241984	
1680	14016000	
32		
6220800		
8640		
4		
6229444	12458888	
8640	1557112000	
8		
623809200		

20.

	$.03\dot{0}$	.3107, approx.
	27	
2700	3000	
90		
1		
2791	2791	
90	209000000	
2		
28830000		

21.

$$\sqrt[3]{\frac{3}{8}}=\frac{\sqrt[3]{3}}{2}=\frac{1.4422}{2}=.7211, \text{ approx.}$$

22.

$$\sqrt[3]{\frac{5}{4}}=\sqrt[3]{\frac{10}{8}}=\frac{\sqrt[3]{10}}{2}=\frac{2.15\dot{4}}{2}=1.077, \text{ approx.}$$

	$1\dot{0}.$	2.154 ...
	8	
1200	2000	
60		
1		
1261	1261	
60	739000	
2		
132300		
3150		
25		
135475	677375	
3150	61625000	
50		
13867500		

23.

$$\sqrt[3]{\frac{7}{27}}=\frac{\sqrt[3]{7}}{3}=\frac{.1913}{3}=.6376, \text{ approx.}$$

$$\begin{array}{l|l}
 & \dot{7}. \quad 1.913, \text{ approx.} \\
 & 1 \\
\hline
300 & 6000 \\
270 & \\
81 & \\
651 & 5859 \\
270 & 141000 \\
162 & \\
108300 & \\
570 & \\
1 & \\
108871 & 108871 \\
570 & 32129000 \\
2 & \\
10944300 &
\end{array}$$

24.

$$\sqrt[3]{\frac{2}{3}}=\sqrt[3]{\frac{18}{27}}=\frac{\sqrt[3]{18}}{3}=\frac{2.6207}{3}$$

$$=.8736, \text{ approx.}$$

$$\begin{array}{l|l}
 & 1\dot{8} \quad 2.6207 \cdots \\
 & 8 \\
\hline
1200 & 10000 \\
360 & \\
36 & \\
1596 & 9576 \\
360 & 424000 \\
72 & \\
202800 & \\
1560 & \\
4 & \\
204364 & 408728 \\
1560 & 15272000000 \\
8 & \\
2059320000 &
\end{array}$$

Art. 214.—Page 179.

1.

$$\begin{array}{l|l}
 & 16x^4-96x^3y+216x^2y^2-216xy^3+81y^4 \;\big|\; 4x^2-12xy+9y^2 \\
 & 16x^4 \\
\hline
8x^2-12xy & -96x^3y \\
 & -96x^3y+144x^2y^2 \\
\hline
8x^2-24xy+9y^2 & 72x^2y^2 \\
 & 72x^2y^2-216xy^3+81y^4
\end{array}$$

$$\sqrt{4x^2-12xy+9y^2}=2x-3y.$$

2.

$$\begin{array}{l|l}
 & x^8-4x^7+10x^6-16x^5+19x^4-16x^3+10x^2-4x+1 \;\big|\; x^4-2x^3+3x^2-2x+1 \\
 & x^8 \\
\hline
2x^4-2x^3 & -4x^7 \\
 & -4x^7+4x^6 \\
\hline
2x^4-4x^3+3x^2 & 6x^6 \\
 & 6x^6-12x^5+9x^4 \\
\hline
2x^4-4x^3+6x^2-2x & -4x^5+10x^4 \\
 & -4x^5+8x^4-12x^3+4x^2 \\
\hline
2x^4-4x^3+6x^2-4x+1 & 2x^4-4x^3+6x^2 \\
 & 2x^4-4x^3+6x^2-4x+1
\end{array}$$

$$
\begin{array}{l|l}
\multicolumn{1}{r}{x^4-2x^3+3x^2-2x+1} & x^2-x+1 \\
\multicolumn{1}{r}{x^4} & \\
\hline
2x^2-x & -2x^3 \\
 & -2x^3+\ \ x^2 \\
\hline
2x^2-2x+1 & 2x^2 \\
 & 2x^2-2x+1 \\
\hline
\end{array}
$$

3.

$$
\begin{array}{l|l}
\multicolumn{1}{r}{x^8-8x^7+16x^6+16x^5-56x^4-32x^3+64x^2+64x+16} & x^4-4x^3+8x+4 \\
\multicolumn{1}{r}{x^8} & \\
\hline
2x^4-4x^3 & -8x^7 \\
 & -8x^7+16x^6 \\
\hline
2x^4-8x^3+8x & 16x^5 \\
 & 16x^5-64x^4\qquad +64x^2 \\
\hline
2x^4-8x^3+16x+4 & 8x^4-32x^3\qquad +64x+16 \\
 & 8x^4-32x^3\qquad +64x+16 \\
\hline
\end{array}
$$

$$
\begin{array}{l|l}
\multicolumn{1}{r}{x^4-4x^3\qquad +8x+4} & x^2-2x-2 \\
\multicolumn{1}{r}{x^4} & \\
\hline
2x^2-2x & -4x^3 \\
 & -4x^3+4x^2 \\
\hline
2x^2-4x-2 & -4x^2 \\
 & -4x^2+8x+4 \\
\hline
\end{array}
$$

4.

$$
\begin{array}{l|l}
\multicolumn{1}{r}{a^{12}-6a^{10}+15a^8-20a^6+15a^4-6a^2+1} & a^6-3a^4+3a^2-1 \\
\multicolumn{1}{r}{a^{12}} & \\
\hline
2a^6-3a^4 & -6a^{10} \\
 & -6a^{10}+\ \ 9a^8 \\
\hline
2a^6-6a^4+3a^2 & 6a^8 \\
 & 6a^8-18a^6+\ \ 9a^4 \\
\hline
2a^6-6a^4+6a^2-1 & -\ \ 2a^6+\ \ 6a^4 \\
 & -\ \ 2a^6+\ \ 6a^4-6a^2+1 \\
\hline
\end{array}
$$

$$
\begin{array}{l|l}
\multicolumn{1}{r}{a^6-3a^4+3a^2-1} & a^2-1 \\
\multicolumn{1}{r}{a^6} & \\
\hline
3a^4-3a^2+1 & -3a^4 \\
 & -3a^4+3a^2-1 \\
\hline
\end{array}
$$

5.

$$
\begin{array}{l|l}
\quad 64x^6+192x^5+240x^4+160x^3+60x^2+12x+1 & 8x^3+12x^2+6x+1 \\
\quad 64x^6 & \\
\hline
\end{array}
$$

$$
\begin{array}{l|l}
16x^3+12x^2 & 192x^5 \\
 & 192x^5+144x^4 \\
\hline
16x^3+24x^2+6x & 96x^4 \\
 & 96x^4+144x^3+36x^2 \\
\hline
16x^3+24x^2+12x+1 & 16x^3+24x^2 \\
 & 16x^3+24x^2+12x+1 \\
\hline
\end{array}
$$

$$
\begin{array}{l|l}
8x^3+12x^2+6x+1 & 2x+1 \\
8x^3 & \\
\hline
\end{array}
$$

$$
\begin{array}{l|l}
12x^2+6x+1 & 12x^2 \\
 & 12x^2+6x+1 \\
\hline
\end{array}
$$

CHAPTER XIX.

Art. 219.—Pages 181, 182.

1. $a^{\frac{1}{4}} = \sqrt[4]{a}$.

2. $b^{\frac{3}{7}} = \sqrt[7]{b^3}$.

3. $2c^{\frac{1}{2}} = 2\sqrt{c}$.

4. $3am^{\frac{5}{4}} = 3a\sqrt[4]{m^5}$.

5. $x^{\frac{3}{4}}y^{\frac{2}{3}} = \sqrt[4]{x^3}\,\sqrt[3]{y^2}$.

6. $m^{\frac{3}{5}}n^{\frac{5}{3}} = \sqrt[5]{m^3}\,\sqrt[3]{n^5}$.

7. $4a^{\frac{m}{5}}b^{\frac{n}{6}} = 4\sqrt[5]{a^m}\,\sqrt[6]{b^n}$.

8. $2c^{\frac{3}{8}}d^{\frac{2}{5}} = 2\sqrt[8]{c^3}\,\sqrt[5]{d^2}$.

9. $5y^{\frac{4}{7}}z^{\frac{9}{2}} = 5\sqrt[7]{y^4}\,\sqrt{z^9}$.

10. $ab^{\frac{1}{3}}c^{\frac{4}{5}}d^{\frac{7}{2}} = a\sqrt[3]{b}\,\sqrt[5]{c^4}\,\sqrt{d^7}$.

11. $\sqrt[5]{x^6} = x^{\frac{6}{5}}$.

12. $\sqrt[3]{y^2} = y^{\frac{2}{3}}$.

13. $\sqrt{n} = n^{\frac{1}{2}}$.

14. $\sqrt[3]{c} = c^{\frac{1}{3}}$.

15. $3\sqrt{m^5} = 3m^{\frac{5}{2}}$.

16. $4\sqrt[7]{a^9} = 4a^{\frac{9}{7}}$.

17. $\sqrt[3]{a^4}\,\sqrt[5]{b} = a^{\frac{4}{3}}b^{\frac{1}{5}}$.

18. $\sqrt{x^5}\,\sqrt[3]{y^2} = x^{\frac{5}{2}}y^{\frac{2}{3}}$.

19. $5\sqrt{m^r}\,\sqrt[3]{n^s} = 5m^{\frac{r}{2}}n^{\frac{s}{3}}$.

20. $2a\sqrt[n]{x}\,\sqrt[n]{y^m} = 2ax^{\frac{1}{n}}y^{\frac{m}{n}}$.

21. $9^{\frac{5}{2}} = (\sqrt{9})^5 = 3^5 = 243$.

22. $27^{\frac{4}{3}} = (\sqrt[3]{27})^4 = 3^4 = 81$.

23. $36^{\frac{3}{2}} = (\sqrt{36})^3 = 6^3 = 216$.

24. $16^{\frac{5}{4}} = (\sqrt[4]{16})^5 = 2^5 = 32$.

25. $(-27)^{\frac{5}{3}} = (\sqrt[3]{-27})^5 = (-3)^5 = -243$.

26. $(-32)^{\frac{4}{5}} = (\sqrt[5]{-32})^4 = (-2)^4 = 16$.

27. $64^{\frac{7}{6}} = (\sqrt[6]{64})^7 = 2^7 = 128$.

28. $(-216)^{\frac{4}{3}} = (\sqrt[3]{-216})^4 = (-6)^4 = 1296$.

Art. 223. — Page 183.

1. $x^3y^{-5}=\frac{x^3}{y^5}$.

2. $x^{-1}y^{\frac{2}{3}}=\frac{y^{\frac{2}{3}}}{x}$.

3. $m^2n^{-\frac{1}{2}}=\frac{m^2}{n^{\frac{1}{2}}}$.

4. $4xy^{-\frac{1}{3}}=\frac{4x}{y^{\frac{1}{3}}}$.

5. $a^{-1}b^{-2}=\frac{1}{ab^2}$.

6. $3a^{\frac{1}{2}}b^{-\frac{2}{5}}=\frac{3a^{\frac{1}{2}}}{b^{\frac{2}{5}}}$.

7. $2x^{-4}y^{-\frac{5}{7}}=\frac{2}{x^4y^{\frac{5}{7}}}$.

8. $a^{-5}b^{-2}c^3=\frac{c^3}{a^5b^2}$.

9. $5a^{-3}b^{-2}c=\frac{5c}{a^3b^2}$.

10. $2m^{-6}n^{-4}=\frac{2}{m^6n^4}$.

11. $3x^{-\frac{2}{3}}y^{-\frac{4}{7}}=\frac{3}{x^{\frac{2}{3}}y^{\frac{4}{7}}}$.

12. $a^{-2}b^{-\frac{3}{4}}c^{-\frac{4}{5}}=\frac{1}{a^2b^{\frac{3}{4}}c^{\frac{4}{5}}}$.

13. $\frac{1}{x}=x^{-1}$.

14. $\frac{a^2}{x^3}=a^2x^{-3}$.

15. $\frac{3}{x^{-2}}=3x^2$.

16. $\frac{1}{2x^{\frac{3}{4}}}=\frac{x^{-\frac{3}{4}}}{2}$.

17. $\frac{3c}{x^2y^{-1}}=3cx^{-2}y$.

18. $\frac{ab^2}{cd^{\frac{1}{2}}}=ab^2c^{-1}d^{-\frac{1}{2}}$.

19. $\frac{5a^2}{2bc^3}=\frac{5a^2b^{-1}c^{-3}}{2}$.

20. $\frac{a^3}{2x^{\frac{2}{3}}y^{\frac{3}{5}}}=\frac{a^3x^{-\frac{2}{3}}y^{-\frac{3}{5}}}{2}$.

21. $\frac{3x}{5m^{-4}n^{-\frac{2}{5}}}=\frac{3m^4n^{\frac{2}{5}}x}{5}$.

22. $\frac{2x^2}{3}=\frac{2}{3x^{-2}}$.

23. $\frac{3x^{\frac{1}{2}}}{4a}=\frac{3}{4ax^{-\frac{1}{2}}}$.

24. $\frac{x^{-3}}{2}=\frac{1}{2x^3}$.

25. $\frac{2c^{-\frac{3}{4}}}{5}=\frac{2}{5c^{\frac{3}{4}}}$.

26. $3a^{\frac{5}{4}}=\frac{3}{a^{-\frac{5}{4}}}$.

27. $\frac{5a^{-2}c}{b^{\frac{1}{2}}}=\frac{5}{a^2b^{\frac{1}{2}}c^{-1}}$.

28. $m^{-\frac{2}{3}}n^{\frac{1}{2}}=\frac{1}{m^{\frac{2}{3}}n^{-\frac{1}{2}}}$.

29. $\frac{x^{-1}y^{\frac{3}{5}}}{z^2}=\frac{1}{xy^{-\frac{3}{5}}z^2}$.

30. $\frac{4a^{-2}b^{-\frac{3}{2}}}{3c^3}=\frac{4}{3a^2b^{\frac{3}{2}}c^3}$.

Art. 224.—Pages 184, 185.

1. $a^3 \times a^{-1} = a^{3-1} = a^2.$

2. $a^2 \times a^{-2} = a^{2-2} = a^0 = 1.$

3. $x^{-1} \times x^{-5} = x^{-1-5} = x^{-6}.$

4. $n^{\frac{3}{4}} \times n^{-\frac{1}{3}} = n^{\frac{3}{4}-\frac{1}{3}} = n^{\frac{5}{12}}.$

5. $2x^{\frac{5}{2}} \times x^{-\frac{3}{2}} = 2x^{\frac{5}{2}-\frac{3}{2}} = 2x.$

6. $3a \times a^{-\frac{2}{3}} = 3a^{1-\frac{2}{3}} = 3a^{\frac{1}{3}}.$

7. $5c^{-3} \times 3c^{-\frac{1}{2}} = 15c^{-3-\frac{1}{2}} = 15c^{-\frac{7}{2}}.$

8. $a^3 \times \sqrt[3]{a^2} = a^3 \times a^{\frac{2}{3}} = a^{3+\frac{2}{3}} = a^{\frac{11}{3}}.$

9. $x^{-1} \times \sqrt[4]{x^{-3}} = x^{-1} \times x^{-\frac{3}{4}} = x^{-1-\frac{3}{4}} = x^{-\frac{7}{4}}.$

10. $m^2 \times \dfrac{4}{\sqrt[5]{m}} = m^2 \times 4m^{-\frac{1}{5}} = 4m^{2-\frac{1}{5}} = 4m^{\frac{9}{5}}.$

11. $2c^{-\frac{2}{7}} \times 3a\sqrt[5]{c^3} = 2c^{-\frac{2}{7}} \times 3ac^{\frac{3}{5}} = 6ac^{-\frac{2}{7}+\frac{3}{5}} = 6ac^{\frac{11}{35}}.$

12. $2a^{-3}b^{\frac{3}{4}} \times ab^{-1} = 2a^{-3+1}b^{\frac{3}{4}-1} = 2a^{-2}b^{-\frac{1}{4}}.$

13. $x^2y^{-\frac{5}{3}} \times \dfrac{x^{-2}y^{\frac{8}{3}}}{2} = \dfrac{x^{2-2}y^{-\frac{5}{3}+\frac{8}{3}}}{2} = \dfrac{y}{2}.$

14. $\sqrt[6]{x} \times 5\sqrt{x^{-5}} = x^{\frac{1}{6}} \times 5x^{-\frac{5}{2}} = 5x^{\frac{1}{6}-\frac{5}{2}} = 5x^{-\frac{7}{3}}.$

15. $\dfrac{1}{a^{\frac{1}{2}}b^{-2}} \times \dfrac{3}{a^{-3}b^{\frac{1}{3}}} = a^{-\frac{1}{2}}b^2 \times 3a^3b^{-\frac{1}{3}} = 3a^{-\frac{1}{2}+3}b^{2-\frac{1}{3}} = 3a^{\frac{5}{2}}b^{\frac{5}{3}}.$

17.

$$\begin{array}{lll}
a^2 - 2 + a^{-2} & & \\
a^2 + 2 + a^{-2} & & \\
\hline
a^4 - 2a^2 + 1 & & \\
\quad\quad 2a^2 - 4 + 2a^{-2} & & \\
\quad\quad\quad\quad 1 - 2a^{-2} + a^{-4} & & \\
\hline
a^4 & -2 & +a^{-4}
\end{array}$$

19.

$$\begin{array}{ll}
x^{-\frac{3}{4}} - x^{-\frac{1}{2}} + x^{-\frac{1}{4}} - 1 & \\
x^{-\frac{1}{4}} + 1 & \\
\hline
x^{-1} - x^{-\frac{3}{4}} + x^{-\frac{1}{2}} - x^{-\frac{1}{4}} & \\
\quad\quad x^{-\frac{3}{4}} - x^{-\frac{1}{2}} + x^{-\frac{1}{4}} - 1 & \\
\hline
x^{-1} & -1
\end{array}$$

18.

$$\begin{array}{ll}
a^{\frac{4}{3}} + a^{\frac{2}{3}}x^{\frac{2}{3}} + x^{\frac{4}{3}} & \\
a^{\frac{2}{3}} - x^{\frac{2}{3}} & \\
\hline
a^2 + a^{\frac{4}{3}}x^{\frac{2}{3}} + a^{\frac{2}{3}}x^{\frac{4}{3}} & \\
\quad\quad - a^{\frac{4}{3}}x^{\frac{2}{3}} - a^{\frac{2}{3}}x^{\frac{4}{3}} - x^2 & \\
\hline
a^2 & -x^2
\end{array}$$

20.

$$\begin{array}{lll}
x^{-2} - 2x^{-1} + 1 - 2x & & \\
x^{-3} + 2x^{-2} & & \\
\hline
x^{-5} - 2x^{-4} + x^{-3} - 2x^{-2} & & \\
\quad\quad 2x^{-4} - 4x^{-3} + 2x^{-2} - 4x^{-1} & & \\
\hline
x^{-5} & -3x^{-3} & -4x^{-1}
\end{array}$$

21.

$$\begin{array}{l}
3a^{-1} - a^{-2}b^{-1} + a^{-3}b^{-2} \\
\underline{6a^3b^2 + 2a^2b + 2a} \\
18a^2b^2 - 6ab + 6 \\
\qquad\quad 6ab - 2 + 2a^{-1}b^{-1} \\
\qquad\qquad\quad 6 - 2a^{-1}b^{-1} + 2a^{-2}b^{-2} \\
\hline
18a^2b^2 \qquad + 10 \qquad\qquad + 2a^{-2}b^{-2}
\end{array}$$

22.

$$\begin{array}{l}
2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 4 + x^{-\frac{1}{3}} \\
\underline{3x^{\frac{4}{3}} + x - 2x^{\frac{2}{3}}} \\
6x^2 - 9x^{\frac{5}{3}} - 12x^{\frac{4}{3}} + 3x \\
\qquad\quad 2x^{\frac{5}{3}} - 3x^{\frac{4}{3}} - 4x + x^{\frac{2}{3}} \\
\qquad\qquad - 4x^{\frac{4}{3}} + 6x + 8x^{\frac{2}{3}} - 2x^{\frac{1}{3}} \\
\hline
6x^2 - 7x^{\frac{5}{3}} - 19x^{\frac{4}{3}} + 5x + 9x^{\frac{2}{3}} - 2x^{\frac{1}{3}}
\end{array}$$

23.

$$\begin{array}{l}
x^{-3}y^2 - x^{-2}y - 2x^{-1} \\
\underline{2x^2y^{-1} + 2x^3y^{-2} - 4x^4y^{-3}} \\
2x^{-1}y - 2 - 4xy^{-1} \\
\qquad\quad 2 - 2xy^{-1} - 4x^2y^{-2} \\
\qquad\quad - 4xy^{-1} + 4x^2y^{-2} + 8x^3y^{-3} \\
\hline
2x^{-1}y \qquad - 10xy^{-1} \qquad\qquad + 8x^3y^{-3}
\end{array}$$

24.

$$\begin{array}{l}
a^{\frac{2}{3}}x^{-\frac{3}{4}} + 2 + a^{-\frac{2}{3}}x^{\frac{3}{4}} \\
\underline{2a^{-\frac{2}{3}}x^{\frac{3}{4}} - 4a^{-\frac{4}{3}}x^{\frac{3}{2}} + 2a^{-2}x^{\frac{9}{4}}} \\
2 + 4a^{-\frac{2}{3}}x^{\frac{3}{4}} + 2a^{-\frac{4}{3}}x^{\frac{3}{2}} \\
\; - 4a^{-\frac{2}{3}}x^{\frac{3}{4}} - 8a^{-\frac{4}{3}}x^{\frac{3}{2}} - 4a^{-2}x^{\frac{9}{4}} \\
\qquad\qquad\quad 2a^{-\frac{4}{3}}x^{\frac{3}{2}} + 4a^{-2}x^{\frac{9}{4}} + 2a^{-\frac{8}{3}}x^3 \\
\hline
2 \qquad\quad - 4a^{-\frac{4}{3}}x^{\frac{3}{2}} \qquad\qquad + 2a^{-\frac{8}{3}}x^3
\end{array}$$

25.

$$
\begin{array}{l}
3a^{\frac{3}{4}}b^{-1} + a^{\frac{1}{4}} - 2a^{-\frac{1}{4}}b \\
\underline{6a^{\frac{1}{4}}b^{-1} - 2a^{-\frac{1}{4}} - 3a^{-\frac{3}{4}}b} \\
18ab^{-2} + 6a^{\frac{1}{2}}b^{-1} - 12 \\
\qquad\quad - 6a^{\frac{1}{2}}b^{-1} - \ 2 + 4a^{-\frac{1}{2}}b \\
\qquad\qquad\qquad\quad \underline{- \ 9 - 3a^{-\frac{1}{2}}b + 6a^{-1}b^{2}} \\
18ab^{-2} \qquad\qquad\quad - 23 + \ a^{-\frac{1}{2}}b + 6a^{-1}b^{2}
\end{array}
$$

Art. 225. — Pages 185, 186.

1. $\dfrac{a^3}{a^{-1}} = a^{3+1} = a^4.$

2. $\dfrac{a}{a^3} = a^{1-3} = a^{-2}.$

3. $\dfrac{a^{\frac{3}{7}}}{a^{\frac{4}{5}}} = a^{\frac{3}{7}-\frac{4}{5}} = a^{-\frac{13}{35}}.$

4. $\dfrac{a^{-\frac{1}{2}}}{a^{-\frac{4}{7}}} = a^{-\frac{1}{2}+\frac{4}{7}} = a^{\frac{1}{14}}.$

5. $\dfrac{3c^{-1}}{\sqrt[4]{c^5}} = \dfrac{3c^{-1}}{c^{\frac{5}{4}}} = 3c^{-1-\frac{5}{4}} = 3c^{-\frac{9}{4}}.$

6. $\dfrac{m^2}{\sqrt[5]{m^{-2}}} = \dfrac{m^2}{m^{-\frac{2}{5}}} = m^{2+\frac{2}{5}} = m^{\frac{12}{5}}.$

7. $x^{\frac{1}{3}} \div \dfrac{1}{\sqrt[4]{x^3}} = x^{\frac{1}{3}} \div x^{-\frac{3}{4}} = x^{\frac{1}{3}+\frac{3}{4}} = x^{\frac{13}{12}}.$

8. $\dfrac{15a}{3a^{-1}\sqrt[3]{b}} = \dfrac{15a}{3a^{-1}b^{\frac{1}{3}}} = 5a^{1+1}b^{-\frac{1}{3}} = 5a^2b^{-\frac{1}{3}}.$

9. $\dfrac{6x^{-1}y^{\frac{2}{3}}}{3xy^{-\frac{1}{5}}} = 2x^{-1-1}y^{\frac{2}{3}+\frac{1}{5}} = 2x^{-2}y^{\frac{13}{15}}.$

11.

$$
\begin{array}{l|l}
a - b & a^{\frac{1}{3}} - b^{\frac{1}{3}} \\
\cline{2-2}
\underline{a - a^{\frac{2}{3}}b^{\frac{1}{3}}} & a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} \\
\quad a^{\frac{2}{3}}b^{\frac{1}{3}} - b & \\
\quad \underline{a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}}} & \\
\qquad\quad a^{\frac{1}{3}}b^{\frac{2}{3}} - b & \\
\qquad\quad \underline{a^{\frac{1}{3}}b^{\frac{2}{3}} - b} &
\end{array}
$$

12.

$$
\begin{array}{l|l}
a^{-2} + 1 & a^{-\frac{2}{3}} + 1 \\
\cline{2-2}
\underline{a^{-2} + a^{-\frac{4}{3}}} & a^{-\frac{4}{3}} - a^{-\frac{2}{3}} + 1 \\
\quad -a^{-\frac{4}{3}} + 1 & \\
\quad \underline{-a^{-\frac{4}{3}} - a^{-\frac{2}{3}}} & \\
\qquad\quad a^{-\frac{2}{3}} + 1 & \\
\qquad\quad \underline{a^{-\frac{2}{3}} + 1} &
\end{array}
$$

13.

$$\begin{array}{l|l}
x^{-2}-5x^{-1}-46-40x & x^{-1}+4 \\ \cline{2-2}
\underline{x^{-2}+4x^{-1}} & x^{-1}-9-10x \\
\quad -9x^{-1} & \\
\quad \underline{-9x^{-1}-36} & \\
\qquad -10 & \\
\qquad \underline{-10-40x} &
\end{array}$$

14.

$$\begin{array}{l|l}
x^{-3}-1 & x^{-\frac{9}{4}}-x^{-\frac{3}{2}}+x^{-\frac{3}{4}}-1 \\ \cline{2-2}
\underline{x^{-3}-x^{-\frac{9}{4}}+x^{-\frac{3}{2}}-x^{-\frac{3}{4}}} & x^{-\frac{3}{4}}+1 \\
\quad x^{-\frac{9}{4}}-x^{-\frac{3}{2}}+x^{-\frac{3}{4}}-1 & \\
\quad \underline{x^{-\frac{9}{4}}-x^{-\frac{3}{2}}+x^{-\frac{3}{4}}-1} &
\end{array}$$

15.

$$\begin{array}{l|l}
m-3m^{\frac{2}{3}}n^{\frac{1}{3}}+3m^{\frac{1}{3}}n^{\frac{2}{3}}-n & m^{\frac{1}{3}}-n^{\frac{1}{3}} \\ \cline{2-2}
\underline{m-\ m^{\frac{2}{3}}n^{\frac{1}{3}}} & m^{\frac{2}{3}}-2m^{\frac{1}{3}}n^{\frac{1}{3}}+n^{\frac{2}{3}} \\
\quad -2m^{\frac{2}{3}}n^{\frac{1}{3}} & \\
\quad \underline{-2m^{\frac{2}{3}}n^{\frac{1}{3}}+2m^{\frac{1}{3}}n^{\frac{2}{3}}} & \\
\qquad m^{\frac{1}{3}}n^{\frac{2}{3}} & \\
\qquad \underline{m^{\frac{1}{3}}n^{\frac{2}{3}}-n} &
\end{array}$$

16.

$$\begin{array}{l|l}
x^{-3}y^{-5}-3x^{-5}y^{-7}+\ x^{-7}y^{-9} & x^{-2}y^{-3}+x^{-3}y^{-4}-x^{-4}y^{-5} \\ \cline{2-2}
\underline{x^{-3}y^{-5}+\ x^{-4}y^{-6}-\ x^{-5}y^{-7}} & x^{-1}y^{-2}-x^{-2}y^{-3}-x^{-3}y^{-4} \\
\quad -\ x^{-4}y^{-6}-2x^{-5}y^{-7}+x^{-7}y^{-9} & \\
\quad \underline{-\ x^{-4}y^{-6}-\ x^{-5}y^{-7}+x^{-6}y^{-8}} & \\
\qquad -\ x^{-5}y^{-7}-x^{-6}y^{-8}+x^{-7}y^{-9} & \\
\qquad \underline{-\ x^{-5}y^{-7}-x^{-6}y^{-8}+x^{-7}y^{-9}} &
\end{array}$$

17.

$$\begin{array}{l|l}
a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b & a^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}} \\ \cline{2-2}
\underline{a-a^{\frac{3}{4}}b^{\frac{1}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}} & a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}} \\
\quad a^{\frac{3}{4}}b^{\frac{1}{4}}+b & \\
\quad \underline{a^{\frac{3}{4}}b^{\frac{1}{4}}-a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{3}{4}}} & \\
\qquad a^{\frac{1}{2}}b^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{3}{4}}+b & \\
\qquad \underline{a^{\frac{1}{2}}b^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{3}{4}}+b} &
\end{array}$$

18.

$$
\begin{array}{l|l}
m^{-\frac{4}{3}}+m^{-\frac{2}{3}}n^{-2}+n^{-4} & m^{-1}+m^{-\frac{2}{3}}n^{-1}+m^{-\frac{1}{3}}n^{-2} \\ \cline{2-2}
\underline{m^{-\frac{4}{3}}+m^{-1}n^{-1}+m^{-\frac{2}{3}}n^{-2}} & m^{-\frac{1}{3}}-n^{-1}+m^{\frac{1}{3}}n^{-2} \\
\quad -m^{-1}n^{-1}+n^{-4} & \\
\quad \underline{-m^{-1}n^{-1}-m^{-\frac{2}{3}}n^{-2}-m^{-\frac{1}{3}}n^{-3}} & \\
\qquad\qquad m^{-\frac{2}{3}}n^{-2}+m^{-\frac{1}{3}}n^{-3}+n^{-4} & \\
\qquad\qquad \underline{m^{-\frac{2}{3}}n^{-2}+m^{-\frac{1}{3}}n^{-3}+n^{-4}} &
\end{array}
$$

Art. 227.—Page 187.

1. $(a^2)^{-3}=a^{-6}$.
2. $(a^{-2})^2=a^{-4}$.
3. $(a^3)^{\frac{5}{2}}=a^{\frac{15}{2}}$.
4. $(c^{-\frac{2}{5}})^{\frac{10}{3}}=c^{-\frac{4}{3}}$.
5. $(x^{-\frac{3}{4}})^{-2}=x^{\frac{3}{2}}$.
6. $(a^{-1})^{\frac{1}{8}}=a^{-\frac{1}{8}}$.
7. $(a^{\frac{1}{3}})^{\frac{6}{5}}=a^{\frac{2}{5}}$.
8. $(\sqrt{x})^{-\frac{1}{3}}=(x^{\frac{1}{2}})^{-\frac{1}{3}}=x^{-\frac{1}{6}}$.
9. $(\sqrt[4]{m^3})^{\frac{2}{3}}=(m^{\frac{3}{4}})^{\frac{2}{3}}=m^{\frac{1}{2}}$.
10. $(\sqrt[5]{y^{-3}})^{-5}=(y^{-\frac{3}{5}})^{-5}=y^3$.
11. $\left(\frac{1}{a^2}\right)^{\frac{3}{2}}=(a^{-2})^{\frac{3}{2}}=a^{-3}$.
12. $(x^{\frac{2n}{3}})^{-\frac{6}{n}}=x^{\frac{2n}{3}\cdot-\frac{6}{n}}=x^{-4}$.
13. $\left(\frac{1}{\sqrt{c}}\right)^{\frac{2}{5}}=(c^{-\frac{1}{2}})^{\frac{2}{5}}=c^{-\frac{1}{5}}$.
14. $\left(\frac{1}{\sqrt[4]{n^3}}\right)^{\frac{4}{3}}=(n^{-\frac{3}{4}})^{\frac{4}{3}}=n^{-1}$.
15. $\sqrt[3]{[(x^{-\frac{1}{2}})^2]}=\sqrt[3]{[x^{-1}]}=x^{-\frac{1}{3}}$.
16. $(a^{1-\frac{n}{m}})^{\frac{1}{m-n}}=a^{\frac{m-n}{m}\cdot\frac{1}{m-n}}=a^{\frac{1}{m}}$.

Art. 229.—Pages 188, 189.

1. $(a^{\frac{2}{3}}-b^{\frac{1}{2}})^2=a^{\frac{4}{3}}-2a^{\frac{2}{3}}b^{\frac{1}{2}}+b$.
2. $(a^{-\frac{3}{2}}+2a)^2=a^{-3}+4a^{-\frac{1}{2}}+4a^2$
3. $(x^{-1}y^2-3x^2y^{-3})^2=x^{-2}y^4-6xy^{-1}+9x^4y^{-6}$.
4. $\sqrt{a^{-2}x^{\frac{3}{4}}}=a^{-1}x^{\frac{3}{8}}$.
5. $\sqrt{9mn^{\frac{1}{3}}}=3m^{\frac{1}{2}}n^{\frac{1}{6}}$.
6. $\sqrt{\frac{c^{\frac{2}{3}}d^{-\frac{5}{4}}}{4xy^3}}=\frac{c^{\frac{1}{3}}d^{-\frac{5}{8}}}{2x^{\frac{1}{2}}y^{\frac{3}{2}}}$.
7. $\sqrt{\frac{a^{-\frac{2}{3}}b^{-1}}{cd^4e^{\frac{1}{2}}}}=\frac{a^{-\frac{1}{3}}b^{-\frac{1}{2}}}{c^{\frac{1}{2}}d^2e^{\frac{1}{4}}}$.

8.

$$\begin{array}{l|l}
\quad 9x^{-4}-12x^{-3}-2x^{-2}+4x^{-1}+1 & 3x^{-2}-2x^{-1}-1 \\
\quad 9x^{-4} & \\
\hline
6x^{-2}-2x^{-1} \;\big|\; -12x^{-3} & \\
\qquad\qquad\qquad\;\; -12x^{-3}+4x^{-2} & \\
\hline
6x^{-2}-4x^{-1}-1 \;\big|\; -6x^{-2} & \\
\qquad\qquad\qquad\;\; \underline{-6x^{-2}+4x^{-1}+1} &
\end{array}$$

9.

$$\begin{array}{l|l}
\quad 4x^{\frac{4}{3}}+4x^{\frac{5}{3}}-15x^{2}-8x^{\frac{7}{3}}+16x^{\frac{8}{3}} & 2x^{\frac{2}{3}}+x-4x^{\frac{4}{3}} \\
\quad 4x^{\frac{4}{3}} & \\
\hline
4x^{\frac{2}{3}}+x \;\big|\; 4x^{\frac{5}{3}} & \\
\qquad\qquad\;\; 4x^{\frac{5}{3}}+\;x^{2} & \\
\hline
4x^{\frac{2}{3}}+2x-4x^{\frac{4}{3}} \;\big|\; -16x^{2} & \\
\qquad\qquad\qquad\qquad \underline{-16x^{2}-8x^{\frac{7}{3}}+16x^{\frac{8}{3}}} &
\end{array}$$

10.

$$\begin{array}{l|l}
\quad a^{3}b^{-\frac{2}{3}}-4a^{\frac{3}{2}}b^{-\frac{1}{3}}+6-4a^{-\frac{3}{2}}b^{\frac{1}{3}}+a^{-3}b^{\frac{2}{3}} & a^{\frac{3}{2}}b^{-\frac{1}{3}}-2+a^{-\frac{3}{2}}b^{\frac{1}{3}} \\
\quad a^{3}b^{-\frac{2}{3}} & \\
\hline
2a^{\frac{3}{2}}b^{-\frac{1}{3}}-2 \;\big|\; -4a^{\frac{3}{2}}b^{-\frac{1}{3}} & \\
\qquad\qquad\qquad\; -4a^{\frac{3}{2}}b^{-\frac{1}{3}}+4 & \\
\hline
2a^{\frac{3}{2}}b^{-\frac{1}{3}}-4+a^{-\frac{3}{2}}b^{\frac{1}{3}} \;\big|\; 2 & \\
\qquad\qquad\qquad\qquad\qquad \underline{2-4a^{-\frac{3}{2}}b^{\frac{1}{3}}+a^{-3}b^{\frac{2}{3}}} &
\end{array}$$

11. $\sqrt[3]{ab^2}=a^{\frac{1}{3}}b^{\frac{2}{3}}.$

12. $\sqrt[3]{-8x^{-4}y^{\frac{2}{3}}}=-2x^{-\frac{4}{3}}y^{\frac{2}{9}}.$

13. $\sqrt[3]{27m^2n^{-\frac{2}{7}}}=3m^{\frac{2}{3}}n^{-\frac{2}{21}}.$

14. $\sqrt[3]{\dfrac{a^{-1}b}{64x^{\frac{1}{3}}}}=\dfrac{a^{-\frac{1}{3}}b^{\frac{1}{3}}}{4x^{\frac{1}{9}}}.$

15.

$$\begin{array}{l|l}
\qquad\qquad\quad 8y^{-2}-12y^{-\frac{11}{6}}+6y^{-\frac{5}{3}}-y^{-\frac{3}{2}} & 2y^{-\frac{2}{3}}-y^{-\frac{1}{2}} \\
\qquad\qquad\quad 8y^{-2} & \\
\hline
12y^{-\frac{4}{3}}-6y^{-\frac{7}{6}}+y^{-1} \;\big|\; -12y^{-\frac{11}{6}} & \\
\qquad\qquad\qquad\qquad\quad \underline{-12y^{-\frac{11}{6}}+6y^{-\frac{5}{3}}-y^{-\frac{3}{2}}} &
\end{array}$$

16.

$$
\begin{array}{l|l}
 & x^{\frac{1}{2}}-9x^{\frac{2}{3}}+33x^{\frac{5}{6}}-63x+66x^{\frac{7}{6}}-36x^{\frac{4}{3}}+8x^{\frac{3}{2}} \;\big|\; x^{\frac{1}{6}}-3x^{\frac{1}{3}}+2x^{\frac{1}{2}} \\
 & x^{\frac{1}{2}} \\
\hline
3x^{\frac{1}{3}}-9x^{\frac{1}{2}}+9x^{\frac{2}{3}} & -9x^{\frac{2}{3}} \\
 & -9x^{\frac{2}{3}}+27x^{\frac{5}{6}}-27x \\
\hline
3x^{\frac{1}{3}}-18x^{\frac{1}{2}}+27x^{\frac{2}{3}} & 6x^{\frac{5}{6}}-36x \\
\quad 6x^{\frac{2}{3}}-18x^{\frac{5}{6}}+4x & \\
\hline
3x^{\frac{1}{3}}-18x^{\frac{1}{2}}+33x^{\frac{2}{3}}-18x^{\frac{5}{6}}+4x & 6x^{\frac{5}{6}}-36x+66x^{\frac{7}{6}}-36x^{\frac{4}{3}}+8x^{\frac{3}{2}} \\
\hline
\end{array}
$$

17. $a^{x-y+2z}a^{2x+y-3z}a^{z}=a^{3x}.$

18. $\dfrac{x^{m+n}x^{m+r}x^{r-m}}{x^{n+2m-r}}=\dfrac{x^{m+n+2r}}{x^{n+2m-r}}=x^{3r-m}.$

19. $(x^{a})^{-b}\div(x^{-a})^{-b}=x^{-ab}\div x^{ab}=x^{-ab-ab}=x^{-2ab}.$

20. $[x^{a^2-ab}x^{b^2-ab}]^{\frac{1}{a-b}}=[x^{a^2-2ab+b^2}]^{\frac{1}{a-b}}=x^{\frac{a^2-2ab+b^2}{a-b}}=x^{a-b}.$

21. $\left(\dfrac{a^{x+y}}{a^{y}}\right)^{x}\div\left(\dfrac{a^{y}}{a^{y-x}}\right)^{x-y}=(a^{x})^{x}\div(a^{x})^{x-y}=a^{x^2}\div a^{x^2-xy}=a^{x^2-x^2+xy}=a^{xy}$

22. $\left[\left(x^{\frac{1}{a-b}}\right)^{a-\frac{b^2}{a}}\right]^{\frac{a}{a+b}}=\left[\left(x^{\frac{1}{a-b}}\right)^{\frac{a^2-b^2}{a}}\right]^{\frac{a}{a+b}}=\left[x^{\frac{a+b}{a}}\right]^{\frac{a}{a+b}}=x.$

23. $$\frac{x^{\frac{1}{2}}(a^{\frac{1}{2}}-x^{\frac{1}{2}})-x^{\frac{3}{2}}(a^{-\frac{1}{2}}-x^{-\frac{1}{2}})}{2(a^{\frac{1}{2}}+x^{\frac{1}{2}})(a^{\frac{1}{2}}-x^{\frac{1}{2}})}=\frac{a^{\frac{1}{2}}x^{\frac{1}{2}}-x-a^{-\frac{1}{2}}x^{\frac{3}{2}}+x}{2(a-x)}$$
$$=\frac{x^{\frac{1}{2}}a^{-\frac{1}{2}}(a-x)}{2(a-x)}=\frac{x^{\frac{1}{2}}}{2a^{\frac{1}{2}}}.$$

24. $$\frac{a-b}{a^{\frac{1}{4}}b^{-\frac{1}{4}}+a^{-\frac{1}{4}}b^{\frac{1}{4}}}=\frac{(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})}{a^{-\frac{1}{4}}b^{-\frac{1}{4}}(a^{\frac{1}{2}}+b^{\frac{1}{2}})}$$
$$=a^{\frac{1}{4}}b^{\frac{1}{4}}(a^{\frac{1}{2}}-b^{\frac{1}{2}})=a^{\frac{3}{4}}b^{\frac{1}{4}}-a^{\frac{1}{4}}b^{\frac{3}{4}}.$$

25. $$\frac{(1-a^{\frac{1}{2}}x^{\frac{1}{2}})^2+(x^{\frac{1}{2}}+a^{\frac{1}{2}})^2}{1-a^{\frac{1}{2}}x^{\frac{1}{2}}+a^{\frac{1}{2}}(a^{\frac{1}{2}}+x^{\frac{1}{2}})}=\frac{1-2a^{\frac{1}{2}}x^{\frac{1}{2}}+ax+x+2a^{\frac{1}{2}}x^{\frac{1}{2}}+a}{1-a^{\frac{1}{2}}x^{\frac{1}{2}}+a+a^{\frac{1}{2}}x^{\frac{1}{2}}}$$
$$=\frac{1+a+x+ax}{1+a}=\frac{(1+a)(1+x)}{1+a}=1+x.$$

26. $(4x^3-3x)(x^2+1)^{-\frac{1}{3}}+3x(x^2+1)^{\frac{2}{3}}=\dfrac{4x^3-3x}{(x^2+1)^{\frac{1}{3}}}+3x(x^2+1)^{\frac{2}{3}}$

$$=\frac{4x^3-3x+3x(x^2+1)}{(x^2+1)^{\frac{1}{3}}}=\frac{4x^3-3x+3x^3+3x}{(x^2+1)^{\frac{1}{3}}}=\frac{7x^3}{(x^2+1)^{\frac{1}{3}}}.$$

27. $\dfrac{(1-3x+x^2)^{\frac{1}{2}}-x(x-3)(1-3x+x^2)^{-\frac{1}{2}}}{1-3x+x^2}$,

multiplying each term by $(1-3x+x^2)^{\frac{1}{2}}$,

$$=\frac{1-3x+x^2-x(x-3)}{(1-3x+x^2)^{\frac{3}{2}}}=\frac{1-3x+x^2-x^2+3x}{(1-3x+x^2)^{\frac{3}{2}}}$$

$$=\frac{1}{(1-3x+x^2)^{\frac{3}{2}}}=(1-3x+x^2)^{-\frac{3}{2}}.$$

28. $\dfrac{m[x^{-\frac{1}{2}}+(m+x)^{-\frac{1}{2}}]}{2[x^{\frac{1}{2}}+(m+x)^{\frac{1}{2}}]}+\dfrac{m+2x}{2x^{\frac{1}{2}}(m+x)^{\frac{1}{2}}}$

$$=\frac{m\left[\dfrac{1}{x^{\frac{1}{2}}}+\dfrac{1}{(m+x)^{\frac{1}{2}}}\right]}{2[x^{\frac{1}{2}}+(m+x)^{\frac{1}{2}}]}+\frac{m+2x}{2x^{\frac{1}{2}}(m+x)^{\frac{1}{2}}}$$

$$=\frac{m\,\dfrac{(m+x)^{\frac{1}{2}}+x^{\frac{1}{2}}}{x^{\frac{1}{2}}(m+x)^{\frac{1}{2}}}}{2[x^{\frac{1}{2}}+(m+x)^{\frac{1}{2}}]}+\frac{m+2x}{2x^{\frac{1}{2}}(m+x)^{\frac{1}{2}}}$$

$$=\frac{m}{2x^{\frac{1}{2}}(m+x)^{\frac{1}{2}}}+\frac{m+2x}{2x^{\frac{1}{2}}(m+x)^{\frac{1}{2}}}=\frac{2m+2x}{2x^{\frac{1}{2}}(m+x)^{\frac{1}{2}}}$$

$$=\frac{m+x}{x^{\frac{1}{2}}(m+x)^{\frac{1}{2}}}=\frac{(m+x)^{\frac{1}{2}}}{x^{\frac{1}{2}}}.$$

29. $\dfrac{x^2+[1+(1+x^2)^{\frac{1}{2}}]^2}{2[1+(1+x^2)^{\frac{1}{2}}]}=\dfrac{x^2+1+2(1+x^2)^{\frac{1}{2}}+(1+x^2)}{2[1+(1+x^2)^{\frac{1}{2}}]}$

$$=\frac{2(1+x^2)^{\frac{1}{2}}+2(1+x^2)}{2[1+(1+x^2)^{\frac{1}{2}}]}=\frac{(1+x^2)^{\frac{1}{2}}+(1+x^2)}{1+(1+x^2)^{\frac{1}{2}}}$$

$$=\frac{(1+x^2)^{\frac{1}{2}}[1+(1+x^2)^{\frac{1}{2}}]}{1+(1+x^2)^{\frac{1}{2}}}=(1+x^2)^{\frac{1}{2}}.$$

CHAPTER XX.

Art. 235. — Page 191.

2. $\sqrt[4]{25}=\sqrt[4]{5^2}=5^{\frac{2}{4}}=5^{\frac{1}{2}}=\sqrt{5}.$

3. $\sqrt[6]{9}=\sqrt[6]{3^2}=3^{\frac{2}{6}}=3^{\frac{1}{3}}=\sqrt[3]{3}.$

4. $\sqrt[9]{8}=\sqrt[9]{2^3}=2^{\frac{3}{9}}=2^{\frac{1}{3}}=\sqrt[3]{2}.$

5. $\sqrt[12]{27}=\sqrt[12]{3^3}=3^{\frac{3}{12}}=3^{\frac{1}{4}}=\sqrt[4]{3}.$

6. $\sqrt[10]{100}=\sqrt[10]{10^2}=10^{\frac{2}{10}}=10^{\frac{1}{5}}=\sqrt[5]{10}.$

7. $\sqrt[12]{81}=\sqrt[12]{3^4}=3^{\frac{4}{12}}=3^{\frac{1}{3}}=\sqrt[3]{3}.$

8. $\sqrt[12]{64}=\sqrt[12]{2^6}=2^{\frac{6}{12}}=2^{\frac{1}{2}}=\sqrt{2}.$

9. $\sqrt[8]{25x^4}=\sqrt[8]{(5x^2)^2}=(5x^2)^{\frac{2}{8}}=(5x^2)^{\frac{1}{4}}=\sqrt[4]{5x^2}.$

10. $\sqrt[10]{32a^5}=\sqrt[10]{(2a)^5}=(2a)^{\frac{5}{10}}=(2a)^{\frac{1}{2}}=\sqrt{2a}.$

11. $\sqrt[4]{49m^4n^6}=\sqrt[4]{(7m^2n^3)^2}=(7m^2n^3)^{\frac{2}{4}}=(7m^2n^3)^{\frac{1}{2}}=\sqrt{7m^2n^3}.$

12. $\sqrt[6]{125a^3b^9}=\sqrt[6]{(5ab^3)^3}=(5ab^3)^{\frac{3}{6}}=(5ab^3)^{\frac{1}{2}}=\sqrt{5ab^3}.$

13. $\sqrt[mn]{a^nb^{2n}}=\sqrt[mn]{(ab^2)^n}=(ab^2)^{\frac{n}{mn}}=(ab^2)^{\frac{1}{m}}=\sqrt[m]{ab^2}.$

Art. 236. — Page 192.

3. $\sqrt{50}=\sqrt{25\times 2}=5\sqrt{2}.$

4. $3\sqrt{24}=3\sqrt{4\times 6}=3\times 2\sqrt{6}=6\sqrt{6}.$

5. $\sqrt{72}=\sqrt{36\times 2}=6\sqrt{2}.$

6. $\sqrt[3]{320}=\sqrt[3]{64\times 5}=4\sqrt[3]{5}.$

7. $2\sqrt[4]{80}=2\sqrt[4]{16\times 5}=2\times 2\sqrt[4]{5}=4\sqrt[4]{5}.$

8. $\sqrt{98a^3b^2}=\sqrt{49a^2b^2\times 2a}=7ab\sqrt{2a}.$

9. $\sqrt[3]{81x^4y^3}=\sqrt[3]{27x^3y^3\times 3x}=3xy\sqrt[3]{3x}.$

10. $7\sqrt{63a^4b^5c^6}=7\sqrt{9a^4b^4c^6\times 7b}=7\times 3a^2b^2c^3\sqrt{7b}=21a^2b^2c^3\sqrt{7b}.$

11. $\sqrt[3]{250x^2y^3z^7}=\sqrt[3]{125y^3z^6\times 2x^2z}=5yz^2\sqrt[3]{2x^2z}.$

12. $\sqrt{25x^3y^4-50x^4y^3}=\sqrt{25x^2y^2\times(xy^2-2x^2y)}=5xy\sqrt{xy^2-2x^2y}.$

13. $\sqrt[3]{54\,a^4b^5 + 135\,a^3b^4} = \sqrt[3]{27\,a^3b^3 \times (2\,ab^2 + 5\,b)} = 3\,ab\,\sqrt[3]{2\,ab^2 + 5\,b}.$

14. $\sqrt{(x^2 - y^2)(x + y)} = \sqrt{(x - y)(x + y)(x + y)}$
$= \sqrt{(x + y)^2 \times (x - y)} = (x + y)\,\sqrt{x - y}.$

15. $\sqrt{ax^2 - 6\,ax + 9\,a} = \sqrt{(x^2 - 6\,x + 9) \times a} = (x - 3)\,\sqrt{a}.$

16. $\sqrt{20\,x^2 + 60\,x + 45} = \sqrt{(4\,x^2 + 12\,x + 9) \times 5} = (2\,x + 3)\sqrt{5}.$

17. $\sqrt{3\,m^3 - 54\,m^2n + 243\,mn^2} = \sqrt{(m^2 - 18\,mn + 81\,n^2) \times 3\,m}$
$= (m - 9\,n)\,\sqrt{3\,m}.$

19. $\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \sqrt{\frac{1}{4} \times 6} = \frac{1}{2}\,\sqrt{6}.$

20. $\sqrt{\frac{5}{6}} = \sqrt{\frac{30}{36}} = \sqrt{\frac{1}{36} \times 30} = \frac{1}{6}\,\sqrt{30}.$

21. $\sqrt{\frac{7}{12}} = \sqrt{\frac{21}{36}} = \sqrt{\frac{1}{36} \times 21} = \frac{1}{6}\,\sqrt{21}.$

22. $\sqrt{\frac{4\,a^2}{27}} = \sqrt{\frac{12\,a^2}{81}} = \sqrt{\frac{4\,a^2}{81} \times 3} = \frac{2\,a}{9}\,\sqrt{3}.$

23. $\sqrt[3]{\frac{3\,x}{4}} = \sqrt[3]{\frac{6\,x}{8}} = \sqrt[3]{\frac{1}{8} \times 6\,x} = \frac{1}{2}\,\sqrt[3]{6\,x}.$

24. $\sqrt[3]{\frac{5}{9}} = \sqrt[3]{\frac{15}{27}} = \sqrt[3]{\frac{1}{27} \times 15} = \frac{1}{3}\,\sqrt[3]{15}.$

25. $\frac{3}{11}\sqrt{\frac{4}{7}} = \frac{3}{11}\sqrt{\frac{28}{49}} = \frac{3}{11}\sqrt{\frac{4}{49} \times 7} = \frac{3}{11} \times \frac{2}{7}\,\sqrt{7} = \frac{6}{77}\,\sqrt{7}.$

26. $\sqrt{\frac{9\,a^2b^3}{10\,cd}} = \sqrt{\frac{90\,a^2b^3cd}{100\,c^2d^2}} = \sqrt{\frac{9\,a^2b^2}{100\,c^2d^2} \times 10\,bcd} = \frac{3\,ab}{10\,cd}\,\sqrt{10\,bcd}.$

27. $\sqrt{\frac{7\,xy^2}{8\,a^5}} = \sqrt{\frac{14\,axy^2}{16\,a^6}} = \sqrt{\frac{y^2}{16\,a^6} \times 14\,ax} = \frac{y}{4\,a^3}\,\sqrt{14\,ax}.$

28. $\sqrt{\frac{ab^2}{4(a + x)}} = \sqrt{\frac{ab^2(a + x)}{4(a + x)^2}} = \sqrt{\frac{b^2}{4(a + x)^2} \times a(a + x)}$
$= \frac{b}{2\,(a + x)}\,\sqrt{a^2 + ax}.$

29. $\frac{a}{a^2 - b^2}\sqrt{\frac{a^3c - 2\,a^2bc + ab^2c}{b^3}} = \frac{a}{a^2 - b^2}\sqrt{\frac{abc\,(a^2 - 2\,ab + b^2)}{b^4}}$
$= \frac{a}{a^2 - b^2}\sqrt{\frac{a^2 - 2\,ab + b^2}{b^4} \times abc} = \frac{a}{a^2 - b^2} \times \frac{a - b}{b^2}\,\sqrt{abc}$
$= \frac{a}{b^2\,(a + b)}\,\sqrt{abc}.$

Art. 237. — Page 193.

2. $3\sqrt{5}=\sqrt{9\times 5}=\sqrt{45}$.

3. $2\sqrt[3]{7}=\sqrt[3]{8\times 7}=\sqrt[3]{56}$.

4. $3\sqrt[4]{2}=\sqrt[4]{81\times 2}=\sqrt[4]{162}$.

5. $4\sqrt[3]{5}=\sqrt[3]{64\times 5}=\sqrt[3]{320}$.

6. $4\sqrt{5ab}=\sqrt{16\times 5ab}=\sqrt{80ab}$.

7. $a^2b\sqrt[3]{ab^2}=\sqrt[3]{a^6b^3\times ab^2}=\sqrt[3]{a^7b^5}$.

8. $5a\sqrt[3]{2x^2}=\sqrt[3]{125a^3\times 2x^2}=\sqrt[3]{250a^3x^2}$.

9. $3mn^3\sqrt[4]{\dfrac{mn^2}{27}}=\sqrt[4]{81m^4n^{12}\times\dfrac{mn^2}{27}}=\sqrt[4]{3m^5n^{14}}$.

10. $(x-1)\sqrt{\dfrac{x+1}{x-1}}=\sqrt{(x-1)^2\dfrac{x+1}{x-1}}=\sqrt{(x-1)(x+1)}=\sqrt{x^2-1}$.

11. $(1+x)\sqrt{\dfrac{2}{1+x}-1}=\sqrt{(1+x)^2\left(\dfrac{2}{1+x}-1\right)}$
$=\sqrt{2(1+x)-(1+x)^2}=\sqrt{2+2x-1-2x-x^2}=\sqrt{1-x^2}$.

12. $\dfrac{1+a}{1-a}\sqrt{\dfrac{1-a}{1+a}}=\sqrt{\dfrac{(1+a)^2(1-a)}{(1-a)^2(1+a)}}=\sqrt{\dfrac{1+a}{1-a}}$.

13. $\dfrac{2x^2-1}{x}\sqrt{\dfrac{1}{(2x^2-1)^2}-1}=\sqrt{\dfrac{(2x^2-1)^2}{x^2}\left(\dfrac{1}{(2x^2-1)^2}-1\right)}$
$=\sqrt{\dfrac{1}{x^2}-\dfrac{(2x^2-1)^2}{x^2}}=\sqrt{\dfrac{1-4x^4+4x^2-1}{x^2}}$
$=\sqrt{\dfrac{4x^2-4x^4}{x^2}}=\sqrt{4-4x^2}$.

Art. 238. — Pages 194, 195.

3. $\sqrt{27}+\sqrt{12}=\sqrt{9\times 3}+\sqrt{4\times 3}=3\sqrt{3}+2\sqrt{3}=5\sqrt{3}$.

4. $\sqrt{96}+\sqrt{54}=\sqrt{16\times 6}+\sqrt{9\times 6}=4\sqrt{6}+3\sqrt{6}=7\sqrt{6}$.

5. $\sqrt{180}-\sqrt{45}=\sqrt{36\times 5}-\sqrt{9\times 5}=6\sqrt{5}-3\sqrt{5}=3\sqrt{5}$.

6. $\sqrt[3]{162}-\sqrt[3]{48}=\sqrt[3]{27\times 6}-\sqrt[3]{8\times 6}=3\sqrt[3]{6}-2\sqrt[3]{6}=\sqrt[3]{6}$.

7. $\sqrt{128}+\sqrt{98}+\sqrt{50}=\sqrt{64\times 2}+\sqrt{49\times 2}+\sqrt{25\times 2}$
$=8\sqrt{2}+7\sqrt{2}+5\sqrt{2}=20\sqrt{2}$.

8. $\sqrt{\frac{16}{15}}-\sqrt{\frac{3}{5}}=\sqrt{\frac{16}{225}\times 15}-\sqrt{\frac{1}{25}\times 15}=\frac{4}{15}\sqrt{15}-\frac{1}{5}\sqrt{15}=\frac{1}{15}\sqrt{15}.$

9. $\sqrt{4a^2b}+\sqrt{9b^3}=\sqrt{4a^2\times b}+\sqrt{9b^2\times b}=2a\sqrt{b}+3b\sqrt{b}$
$=(2a+3b)\sqrt{b}.$

10. $\sqrt{75}+\sqrt{48}-\sqrt{245}=\sqrt{25\times 3}+\sqrt{16\times 3}-\sqrt{49\times 5}$
$=5\sqrt{3}+4\sqrt{3}-7\sqrt{5}=9\sqrt{3}-7\sqrt{5}.$

11. $\sqrt[3]{16}+\sqrt[3]{54}+\sqrt[3]{128}=\sqrt[3]{8\times 2}+\sqrt[3]{27\times 2}+\sqrt[3]{64\times 2}$
$=2\sqrt[3]{2}+3\sqrt[3]{2}+4\sqrt[3]{2}=9\sqrt[3]{2}.$

12. $\sqrt{\frac{5}{9}}-\sqrt{\frac{1}{5}}+\sqrt{\frac{1}{45}}=\sqrt{\frac{1}{9}\times 5}-\sqrt{\frac{1}{25}\times 5}+\sqrt{\frac{1}{225}\times 5}$
$=\frac{1}{3}\sqrt{5}-\frac{1}{5}\sqrt{5}+\frac{1}{15}\sqrt{5}=\frac{1}{5}\sqrt{5}.$

13. $\sqrt{\frac{3}{8}}-\sqrt{\frac{1}{6}}+\sqrt{\frac{2}{27}}=\sqrt{\frac{1}{16}\times 6}-\sqrt{\frac{1}{36}\times 6}+\sqrt{\frac{1}{81}\times 6}$
$=\frac{1}{4}\sqrt{6}-\frac{1}{6}\sqrt{6}+\frac{1}{9}\sqrt{6}=\frac{7}{36}\sqrt{6}.$

14. $\sqrt[3]{\frac{1}{4}}+\sqrt[3]{\frac{1}{32}}+\sqrt[3]{\frac{2}{3}}=\sqrt[3]{\frac{1}{8}\times 2}+\sqrt[3]{\frac{1}{64}\times 2}+\sqrt[3]{\frac{1}{27}\times 18}$
$=\frac{1}{2}\sqrt[3]{2}+\frac{1}{4}\sqrt[3]{2}+\frac{1}{3}\sqrt[3]{18}=\frac{3}{4}\sqrt[3]{2}+\frac{1}{3}\sqrt[3]{18}.$

15. $7\sqrt{27}-\sqrt{75}-24\sqrt{\frac{1}{12}}-27\sqrt{\frac{1}{27}}$
$=7\sqrt{9\times 3}-\sqrt{25\times 3}-24\sqrt{\frac{1}{36}\times 3}-27\sqrt{\frac{1}{81}\times 3}$
$=21\sqrt{3}-5\sqrt{3}-\frac{24}{6}\sqrt{3}-\frac{27}{9}\sqrt{3}$
$=21\sqrt{3}-5\sqrt{3}-4\sqrt{3}-3\sqrt{3}=9\sqrt{3}.$

16. $\sqrt{27ab^2}+\sqrt{75a^3}+(a-3b)\sqrt{3a}$
$=\sqrt{9b^2\times 3a}+\sqrt{25a^2\times 3a}+(a-3b)\sqrt{3a}$
$=3b\sqrt{3a}+5a\sqrt{3a}+(a-3b)\sqrt{3a}$
$=(3b+5a+a-3b)\sqrt{3a}=6a\sqrt{3a}.$

17. $\sqrt{9a^5+18a^4b}-\sqrt{4ab^6+8b^7}=\sqrt{9a^4(a+2b)}-\sqrt{4b^6(a+2b)}$
$=3a^2\sqrt{a+2b}-2b^3\sqrt{a+2b}=(3a^2-2b^3)\sqrt{a+2b}.$

18. $\sqrt[3]{24}+5\sqrt[3]{54}-\sqrt[3]{250}-\sqrt[3]{192}$
$$=\sqrt[3]{8\times 3}+5\sqrt[3]{27\times 2}-\sqrt[3]{125\times 2}-\sqrt[3]{64\times 3}$$
$$=2\sqrt[3]{3}+15\sqrt[3]{2}-5\sqrt[3]{2}-4\sqrt[3]{3}=10\sqrt[3]{2}-2\sqrt[3]{3}.$$

19. $\sqrt{28a^2x-28ax+7x}-\sqrt{7a^2x+42ax+63x}$
$$=\sqrt{(4a^2-4a+1)\times 7x}-\sqrt{(a^2+6a+9)\times 7x}$$
$$=(2a-1)\sqrt{7x}-(a+3)\sqrt{7x}$$
$$=(2a-1-a-3)\sqrt{7x}=(a-4)\sqrt{7x}.$$

20. $x\sqrt{\dfrac{x-y}{x+y}}+y\sqrt{\dfrac{x+y}{x-y}}-\dfrac{3y^2-x^2}{x^2-y^2}\sqrt{x^2-y^2}$
$$=x\sqrt{\frac{x^2-y^2}{(x+y)^2}}+y\sqrt{\frac{x^2-y^2}{(x-y)^2}}-\frac{3y^2-x^2}{x^2-y^2}\sqrt{x^2-y^2}$$
$$=\frac{x}{x+y}\sqrt{x^2-y^2}+\frac{y}{x-y}\sqrt{x^2-y^2}-\frac{3y^2-x^2}{x^2-y^2}\sqrt{x^2-y^2}$$
$$=\frac{x(x-y)+y(x+y)-(3y^2-x^2)}{x^2-y^2}\sqrt{x^2-y^2}$$
$$=\frac{x^2-xy+xy+y^2-3y^2+x^2}{x^2-y^2}\sqrt{x^2-y^2}$$
$$=\frac{2x^2-2y^2}{x^2-y^2}\sqrt{x^2-y^2}=2\sqrt{x^2-y^2}.$$

Art. 239. — Pages 195, 196.

2. $\sqrt[3]{2}=2^{\frac{1}{3}}=2^{\frac{2}{6}}=\sqrt[6]{2^2}=\sqrt[6]{4}.$

$\sqrt{3}=3^{\frac{1}{2}}=3^{\frac{3}{6}}=\sqrt[6]{3^3}=\sqrt[6]{27}.$

3. $\sqrt{5}=5^{\frac{1}{2}}=5^{\frac{3}{6}}=\sqrt[6]{5^3}=\sqrt[6]{125}.$

$\sqrt[3]{4}=4^{\frac{1}{3}}=4^{\frac{2}{6}}=\sqrt[6]{4^2}=\sqrt[6]{16}.$

$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{2}{6}}=\sqrt[6]{3^2}=\sqrt[6]{9}.$

4. $\sqrt[3]{5}=5^{\frac{1}{3}}=5^{\frac{4}{12}}=\sqrt[12]{5^4}=\sqrt[12]{625}.$

$\sqrt[4]{6}=6^{\frac{1}{4}}=6^{\frac{3}{12}}=\sqrt[12]{6^3}=\sqrt[12]{216}.$

$\sqrt[6]{7}=7^{\frac{1}{6}}=7^{\frac{2}{12}}=\sqrt[12]{7^2}=\sqrt[12]{49}.$

5. $\sqrt[3]{2a}=(2a)^{\frac{1}{3}}=(2a)^{\frac{5}{15}}=\sqrt[15]{(2a)^5}=\sqrt[15]{32a^5}.$

$\sqrt[5]{3b}=(3b)^{\frac{1}{5}}=(3b)^{\frac{3}{15}}=\sqrt[15]{(3b)^3}=\sqrt[15]{27b^3}.$

$\sqrt[5]{4c}=(4c)^{\frac{1}{5}}=(4c)^{\frac{3}{15}}=\sqrt[15]{(4c)^3}=\sqrt[15]{64c^3}.$

6. $\sqrt[6]{xy}=(xy)^{\frac{1}{6}}=(xy)^{\frac{4}{24}}=\sqrt[24]{(xy)^4}=\sqrt[24]{x^4y^4}.$

$\sqrt[4]{yz}=(yz)^{\frac{1}{4}}=(yz)^{\frac{6}{24}}=\sqrt[24]{(yz)^6}=\sqrt[24]{y^6z^6}.$

$\sqrt[8]{zx}=(zx)^{\frac{1}{8}}=(zx)^{\frac{3}{24}}=\sqrt[24]{(zx)^3}=\sqrt[24]{z^3x^3}.$

7. $\sqrt[6]{a+b}=(a+b)^{\frac{1}{6}}=(a+b)^{\frac{2}{12}}=\sqrt[12]{(a+b)^2}=\sqrt[12]{a^2+2ab+b^2}.$

$\sqrt[4]{a-b}=(a-b)^{\frac{1}{4}}=(a-b)^{\frac{3}{12}}=\sqrt[12]{(a-b)^3}=\sqrt[12]{a^3-3a^2b+3ab^2-b^3}.$

8. $\sqrt[3]{2}=2^{\frac{1}{3}}=2^{\frac{5}{15}}=\sqrt[15]{2^5}=\sqrt[15]{32}.$

$\sqrt[5]{3}=3^{\frac{1}{5}}=3^{\frac{3}{15}}=\sqrt[15]{3^3}=\sqrt[15]{27}.$

$\therefore$ $\sqrt[3]{2}$ is greater than $\sqrt[5]{3}$.

9. $\sqrt[4]{3}=3^{\frac{1}{4}}=3^{\frac{5}{20}}=\sqrt[20]{3^5}=\sqrt[20]{243}.$

$\sqrt[5]{5}=5^{\frac{1}{5}}=5^{\frac{4}{20}}=\sqrt[20]{5^4}=\sqrt[20]{625}.$

$\therefore$ $\sqrt[5]{5}$ is greater than $\sqrt[4]{3}$.

10. $\sqrt{3}=3^{\frac{1}{2}}=3^{\frac{6}{12}}=\sqrt[12]{3^6}=\sqrt[12]{729}.$

$\sqrt[3]{4}=4^{\frac{1}{3}}=4^{\frac{4}{12}}=\sqrt[12]{4^4}=\sqrt[12]{256}.$

$\sqrt[4]{7}=7^{\frac{1}{4}}=7^{\frac{3}{12}}=\sqrt[12]{7^3}=\sqrt[12]{343}.$

$\therefore$ $\sqrt{3}$ is the greatest, $\sqrt[4]{7}$ is next in magnitude, and $\sqrt[3]{4}$ is the least.

Art. 240. — Pages 196–198.

3. $\sqrt{6}\times\sqrt{42}=\sqrt{252}=\sqrt{36\times7}=6\sqrt{7}.$

4. $5\sqrt{10}\times3\sqrt{15}=15\sqrt{150}=15\sqrt{25\times6}=15\times5\sqrt{6}=75\sqrt{6}.$

5. $2\sqrt{3x}\times5\sqrt{15x}=10\sqrt{45x^2}=10\sqrt{9x^2\times5}=10\times3x\sqrt{5}=30x\sqrt{5}.$

6. $\sqrt[3]{a^2b}\times\sqrt[3]{abc^2}=\sqrt[3]{a^3b^2c^2}=\sqrt[3]{a^3\times b^2c^2}=a\sqrt[3]{b^2c^2}.$

7. $\frac{3}{4}\sqrt[3]{12}\times\frac{2}{3}\sqrt[3]{2}=\frac{3}{4}\times\frac{2}{3}\sqrt[3]{24}=\frac{1}{2}\sqrt[3]{8\times3}=\frac{1}{2}\times2\sqrt[3]{3}=\sqrt[3]{3}.$

8. $\sqrt[3]{2}=2^{\frac{1}{3}}=2^{\frac{4}{12}}=\sqrt[12]{16}$; $\sqrt[4]{3}=3^{\frac{1}{4}}=3^{\frac{3}{12}}=\sqrt[12]{27}.$

$\therefore$ $\sqrt[3]{2}\times\sqrt[4]{3}=\sqrt[12]{16}\times\sqrt[12]{27}=\sqrt[12]{432}.$

9. $\sqrt{ax}=(ax)^{\frac{1}{2}}=(ax)^{\frac{3}{6}}=\sqrt[6]{a^3x^3}$; $\sqrt[3]{bx}=(bx)^{\frac{1}{3}}=(bx)^{\frac{2}{6}}=\sqrt[6]{b^2x^2}.$

$\therefore$ $\sqrt{ax}\times\sqrt[3]{bx}=\sqrt[6]{a^3x^3}\times\sqrt[6]{b^2x^2}=\sqrt[6]{a^3b^2x^5}.$

10. $\sqrt[3]{4a^2}=(4a^2)^{\frac{1}{3}}=(4a^2)^{\frac{2}{6}}=\sqrt[6]{16a^4}$; $\sqrt{2a}=(2a)^{\frac{1}{2}}=(2a)^{\frac{3}{6}}=\sqrt[6]{8a^3}.$

$\therefore$ $\sqrt[3]{4a^2}\times\sqrt{2a}=\sqrt[6]{16a^4}\times\sqrt[6]{8a^3}=\sqrt[6]{128a^7}=\sqrt[6]{64a^6\times2a}$

$=2a\sqrt[6]{2a}.$

11. $\sqrt[5]{3}=3^{\frac{1}{5}}=3^{\frac{2}{10}}=\sqrt[10]{9}$; $\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{5}{10}}=\sqrt[10]{32}$.

$\therefore\ 4\sqrt[5]{3}\times 3\sqrt{2}=4\sqrt[10]{9}\times 3\sqrt[10]{32}=12\sqrt[10]{288}$.

12. $\sqrt[4]{xy}\times\sqrt[4]{yz}\times\sqrt[4]{zx}=\sqrt[4]{x^2y^2z^2}=(xyz)^{\frac{2}{4}}=(xyz)^{\frac{1}{2}}=\sqrt{xyz}$.

13. $\sqrt{3}=3^{\frac{1}{2}}=3^{\frac{3}{6}}=\sqrt[6]{27}$; $\sqrt[3]{2}=2^{\frac{1}{3}}=2^{\frac{2}{6}}=\sqrt[6]{4}$;

$$\sqrt[3]{\frac{1}{6}}=\left(\frac{1}{6}\right)^{\frac{1}{3}}=\left(\frac{1}{6}\right)^{\frac{2}{6}}=\sqrt[6]{\frac{1}{36}}.$$

$$\therefore\ \sqrt{3}\times\sqrt[3]{2}\times\sqrt[3]{\frac{1}{6}}=\sqrt[6]{27}\times\sqrt[6]{4}\times\sqrt[6]{\frac{1}{36}}=\sqrt[6]{\left(27\times 4\times\frac{1}{36}\right)}=\sqrt[6]{3}.$$

14. $\sqrt[5]{2x}=(2x)^{\frac{1}{5}}=(2x)^{\frac{3}{15}}=\sqrt[15]{8x^3}$; $\sqrt[3]{3x}=(3x)^{\frac{1}{3}}=(3x)^{\frac{5}{15}}=\sqrt[15]{243x^5}$;

$$\sqrt[5]{\frac{1}{3x^2}}=\left(\frac{1}{3x^2}\right)^{\frac{1}{5}}=\left(\frac{1}{3x^2}\right)^{\frac{3}{15}}=\sqrt[15]{\frac{1}{27x^6}}.$$

$$\therefore\ \sqrt[5]{2x}\times\sqrt[3]{3x}\times\sqrt[5]{\frac{1}{3x^2}}=\sqrt[15]{8x^3}\times\sqrt[15]{243x^5}\times\sqrt[15]{\frac{1}{27x^6}}$$

$$=\sqrt[15]{\left(8x^3\times 243x^5\times\frac{1}{27x^6}\right)}=\sqrt[15]{72x^2}.$$

17.

$$\begin{array}{r}
\sqrt{x}-2 \\
\underline{\sqrt{x}+3} \\
x-2\sqrt{x} \qquad \\
\underline{3\sqrt{x}-6} \\
x+\ \sqrt{x}-6
\end{array}$$

18.

$$\begin{array}{r}
\sqrt{5}-3\sqrt{2} \\
\underline{2\sqrt{5}+\ \sqrt{2}} \\
10-6\sqrt{10} \qquad \\
\underline{\sqrt{10}-6} \\
4-5\sqrt{10} \qquad
\end{array}$$

19.

$$\begin{array}{r}
\sqrt{x}-4\sqrt{3} \\
\underline{2\sqrt{x}+\ \sqrt{3}} \\
2x-8\sqrt{3x} \qquad \\
\underline{\sqrt{3x}-12} \\
2x-7\sqrt{3x}-12
\end{array}$$

20.

$$\begin{array}{r}
2\sqrt{a}-3\sqrt{b} \\
\underline{4\sqrt{a}+\ \sqrt{b}} \\
8a-12\sqrt{ab} \qquad \\
\underline{2\sqrt{ab}-3b} \\
8a-10\sqrt{ab}-3b
\end{array}$$

21.

$$\begin{array}{l}
\sqrt{x}-\sqrt{y}+\sqrt{z} \\
\underline{\sqrt{x}+\sqrt{y}-\sqrt{z}} \\
x-\sqrt{xy}+\sqrt{xz} \\
\qquad \sqrt{xy} \qquad -y+\ \sqrt{yz} \\
\qquad\quad -\sqrt{xz} \qquad +\ \sqrt{yz}-z \\
\hline
x \qquad\qquad\qquad -y+2\sqrt{yz}-z
\end{array}$$

22.

$$\begin{array}{r}
\sqrt{x+1}-2\sqrt{x} \\
\underline{2\sqrt{x+1}+\ \sqrt{x}} \\
2(x+1)-4\sqrt{x^2+x} \\
\underline{\sqrt{x^2+x}-2x} \\
2x+2-3\sqrt{x^2+x}-2x \\
=2-3\sqrt{x^2+x}
\end{array}$$

23.

$$\begin{array}{l}
\sqrt{2}-\sqrt{3}+\sqrt{5}\\
\underline{\sqrt{2}+\sqrt{3}+\sqrt{5}}\\
\quad 2-\sqrt{6}+\sqrt{10}\\
-3+\sqrt{6}\qquad\quad+\sqrt{15}\\
\underline{\quad 5\qquad+\sqrt{10}-\sqrt{15}}\\
\quad 4\qquad+2\sqrt{10}
\end{array}$$

24.

$$\begin{array}{l}
3\sqrt{5}-2\sqrt{6}+\sqrt{7}\\
\underline{6\sqrt{5}+4\sqrt{6}-2\sqrt{7}}\\
\quad 90-12\sqrt{30}+6\sqrt{35}\\
-48+12\sqrt{30}\qquad\quad+4\sqrt{42}\\
\underline{-14\qquad\qquad-6\sqrt{35}+4\sqrt{42}}\\
\quad 28\qquad\qquad\qquad+8\sqrt{42}
\end{array}$$

25.

$$\begin{array}{l}
8\sqrt{3}+10\sqrt{2}-3\sqrt{5}\\
\underline{4\sqrt{3}-\;5\sqrt{2}-\;\sqrt{5}}\\
\quad 96+40\sqrt{6}-12\sqrt{15}\\
-100-40\sqrt{6}\qquad\quad+15\sqrt{10}\\
\underline{\quad 15\qquad\quad-\;8\sqrt{15}-10\sqrt{10}}\\
\quad 11\qquad\quad-20\sqrt{15}+\;5\sqrt{10}
\end{array}$$

26. $(2\sqrt{3}-3)^2=12-12\sqrt{3}+9=21-12\sqrt{3}.$

27. $(3\sqrt{8}+5\sqrt{3})^2=72+30\sqrt{24}+75=147+30\sqrt{4\times 6}=147+60\sqrt{6}.$

28. $(\sqrt{1-a^2}+a)^2=1-a^2+2a\sqrt{1-a^2}+a^2=1+2a\sqrt{1-a^2}.$

29. $(\sqrt{a+b}-\sqrt{a-b})^2=a+b-2\sqrt{a^2-b^2}+a-b=2a-2\sqrt{a^2-b^2}.$

30. $(\sqrt{x^2+1}+x)(\sqrt{x^2+1}-x)=x^2+1-x^2=1.$

31. $(\sqrt{x+1}+\sqrt{x-1})(\sqrt{x+1}-\sqrt{x-1})=x+1-(x-1)=2.$

32. $(3\sqrt{2x+5}+2\sqrt{3x-1})(3\sqrt{2x+5}-2\sqrt{3x-1})$

$$=9(2x+5)-4(3x-1)=18x+45-12x+4=6x+49.$$

Art. 241. — Page 199.

2. $\dfrac{\sqrt{108}}{\sqrt{6}}=\sqrt{18}=\sqrt{9\times 2}=3\sqrt{2}.$

3. $\dfrac{\sqrt{50c^3}}{\sqrt{2c}}=\sqrt{25c^2}=5c.$

4. $\dfrac{\sqrt[3]{9a^4}}{\sqrt[3]{3a}}=\sqrt[3]{3a^3}=a\sqrt[3]{3}.$

5. $\sqrt{6}=6^{\frac{1}{2}}=6^{\frac{3}{6}}=\sqrt[6]{216}.$

$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{2}{6}}=\sqrt[6]{9}.$

$\therefore\ \dfrac{\sqrt{6}}{\sqrt[3]{3}}=\dfrac{\sqrt[6]{216}}{\sqrt[6]{9}}=\sqrt[6]{24}.$

6. $\sqrt[3]{18}=18^{\frac{1}{3}}=18^{\frac{2}{6}}=\sqrt[6]{324}.$

$\sqrt{6}\ =6^{\frac{1}{2}}=6^{\frac{3}{6}}=\sqrt[6]{216}.$

$\therefore\ \dfrac{\sqrt[3]{18}}{\sqrt{6}}=\dfrac{\sqrt[6]{324}}{\sqrt[6]{216}}=\sqrt[6]{\dfrac{324}{216}}=\sqrt[6]{\dfrac{3}{2}}.$

7. $\sqrt[5]{2} = 2^{\frac{1}{5}} = 2^{\frac{4}{20}} = \sqrt[20]{16}.$

$\sqrt[4]{3} = 3^{\frac{1}{4}} = 3^{\frac{5}{20}} = \sqrt[20]{243}.$

$\therefore \dfrac{\sqrt[5]{2}}{\sqrt[4]{3}} = \dfrac{\sqrt[20]{16}}{\sqrt[20]{243}} = \sqrt[20]{\dfrac{16}{243}}.$

8. $\sqrt[5]{12} = 12^{\frac{1}{5}} = 12^{\frac{3}{15}} = \sqrt[15]{1728}.$

$\sqrt[3]{2} = 2^{\frac{1}{3}} = 2^{\frac{5}{15}} = \sqrt[15]{32}.$

$\therefore \dfrac{\sqrt[5]{12}}{\sqrt[3]{2}} = \dfrac{\sqrt[15]{1728}}{\sqrt[15]{32}} = \sqrt[15]{54}.$

9. $\sqrt[3]{4a} = (4a)^{\frac{1}{3}} = (4a)^{\frac{4}{12}} = \sqrt[12]{256a^4}.$

$\sqrt[4]{2a} = (2a)^{\frac{1}{4}} = (2a)^{\frac{3}{12}} = \sqrt[12]{8a^3}.$

$\therefore \dfrac{\sqrt[3]{4a}}{\sqrt[4]{2a}} = \dfrac{\sqrt[12]{256a^4}}{\sqrt[12]{8a^3}} = \sqrt[12]{32a}.$

10. $\sqrt[3]{3a^2b} = (3a^2b)^{\frac{1}{3}} = (3a^2b)^{\frac{5}{15}} = \sqrt[15]{243a^{10}b^5}.$

$\sqrt[5]{6a^3b^2} = (6a^3b^2)^{\frac{1}{5}} = (6a^3b^2)^{\frac{3}{15}} = \sqrt[15]{216a^9b^6}.$

$\therefore \dfrac{\sqrt[3]{3a^2b}}{\sqrt[5]{6a^3b^2}} = \dfrac{\sqrt[15]{243a^{10}b^5}}{\sqrt[15]{216a^9b^6}} = \sqrt[15]{\dfrac{9a}{8b}}.$

11. $\sqrt[6]{12x^3y^2z^5} = (12x^3y^2z^5)^{\frac{1}{6}} = (12x^3y^2z^5)^{\frac{2}{12}} = \sqrt[12]{144x^6y^4z^{10}}.$

$\sqrt[4]{2x^2yz^3} = (2x^2yz^3)^{\frac{1}{4}} = (2x^2yz^3)^{\frac{3}{12}} = \sqrt[12]{8x^6y^3z^9}.$

$\therefore \dfrac{\sqrt[6]{12x^3y^2z^5}}{\sqrt[4]{2x^2yz^3}} = \dfrac{\sqrt[12]{144x^6y^4z^{10}}}{\sqrt[12]{8x^6y^3z^9}} = \sqrt[12]{18yz}.$

Art. 242.—Pages 199, 200.

3. $(\sqrt[4]{5})^3 = \sqrt[4]{125}.$ 4. $(\sqrt[4]{7})^2 = \sqrt{7}.$

5. $(\sqrt[3]{a^2x})^5 = \sqrt[3]{a^{10}x^5} = \sqrt[3]{a^9x^3 \times ax^2} = a^3x\sqrt[3]{ax^2}.$

6. $(\sqrt[6]{18})^3 = \sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}.$ 7. $(\sqrt[8]{a-b})^4 = \sqrt{a-b}.$

8. $(4\sqrt{3x})^3 = 64\sqrt{27x^3} = 64\sqrt{9x^2 \times 3x} = 64 \times 3x\sqrt{3x} = 192x\sqrt{3x}$

9. $(\sqrt[12]{32})^3 = \sqrt[4]{32} = \sqrt[4]{16 \times 2} = 2\sqrt[4]{2}.$

10. $(3a\sqrt[3]{bx})^4 = 81a^4\sqrt[3]{b^4x^4} = 81a^4\sqrt[3]{b^3x^3 \times bx} = 81a^4 \times bx\sqrt[3]{bx}$
$= 81a^4bx\sqrt[3]{bx}.$

11. $(3\sqrt[6]{24a^4b^5})^2 = 9\sqrt[3]{24a^4b^5} = 9\sqrt[3]{8a^3b^3 \times 3ab^2} = 9 \times 2ab\sqrt[3]{3ab^2}$
$= 18ab\sqrt[3]{3ab^2}.$

14. $\sqrt{(\sqrt{2})} = \sqrt[4]{2}.$ 16. $\sqrt[5]{(\sqrt{32})} = \sqrt{2}.$

15. $\sqrt[3]{(\sqrt{125})} = \sqrt{5}.$ 17. $\sqrt[3]{(\sqrt[5]{27a^3})} = \sqrt[5]{3a}.$

18. $\sqrt[3]{(\sqrt[4]{a+b})} = \sqrt[12]{a+b}.$

19. $\sqrt{(\sqrt[3]{x^2-2x+1})} = \sqrt[3]{x-1}.$

20. $\sqrt[3]{(3\sqrt{3})} = \sqrt[3]{(\sqrt{9\times 3})} = \sqrt[3]{(\sqrt{27})} = \sqrt{3}.$

21. $\sqrt[4]{(\sqrt[5]{x^8y^{12}})} = \sqrt[5]{x^2y^3}.$

22. $\sqrt[5]{(4\sqrt{2})} = \sqrt[5]{(\sqrt{16\times 2})} = \sqrt[5]{(\sqrt{32})} = \sqrt{2}.$

Art. 243. — Page 201.

2. $\dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{\sqrt{2}\sqrt{2}} = \dfrac{3\sqrt{2}}{2}.$

3. $\dfrac{1}{\sqrt[3]{4}} = \dfrac{\sqrt[3]{2}}{\sqrt[3]{4}\sqrt[3]{2}} = \dfrac{\sqrt[3]{2}}{\sqrt[3]{8}} = \dfrac{\sqrt[3]{2}}{2}.$

4. $\dfrac{2x}{\sqrt{5xy}} = \dfrac{2x\sqrt{5xy}}{\sqrt{5xy}\sqrt{5xy}} = \dfrac{2x\sqrt{5xy}}{5xy} = \dfrac{2}{5y}\sqrt{5xy}.$

5. $\dfrac{5}{\sqrt[3]{9a^2}} = \dfrac{5\sqrt[3]{3a}}{\sqrt[3]{9a^2}\sqrt[3]{3a}} = \dfrac{5\sqrt[3]{3a}}{\sqrt[3]{27a^3}} = \dfrac{5}{3a}\sqrt[3]{3a}.$

6. $\dfrac{1}{\sqrt[5]{16x^3}} = \dfrac{\sqrt[5]{2x^2}}{\sqrt[5]{16x^3}\sqrt[5]{2x^2}} = \dfrac{\sqrt[5]{2x^2}}{\sqrt[5]{32x^5}} = \dfrac{1}{2x}\sqrt[5]{2x^2}.$

7. $\dfrac{2c}{\sqrt[4]{27a^2}} = \dfrac{2c\sqrt[4]{3a^2}}{\sqrt[4]{27a^2}\sqrt[4]{3a^2}} = \dfrac{2c\sqrt[4]{3a^2}}{\sqrt[4]{81a^4}} = \dfrac{2c\sqrt[4]{3a^2}}{3a}.$

Art. 244. — Page 202.

3. $\dfrac{4}{3+\sqrt{2}} = \dfrac{4(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})} = \dfrac{4(3-\sqrt{2})}{9-2} = \dfrac{12-4\sqrt{2}}{7}.$

4. $\dfrac{4-\sqrt{3}}{2-\sqrt{3}} = \dfrac{(4-\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} = \dfrac{8+2\sqrt{3}-3}{4-3} = 5+2\sqrt{3}.$

5. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}} = \dfrac{(\sqrt{2}-\sqrt{3})^2}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = \dfrac{2-2\sqrt{6}+3}{2-3} = \dfrac{5-2\sqrt{6}}{-1}$
$= 2\sqrt{6}-5.$

6. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \dfrac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})} = \dfrac{a+2\sqrt{ab}+b}{a-b}.$

7. $\dfrac{2\sqrt{5}+\sqrt{2}}{\sqrt{5}-3\sqrt{2}} = \dfrac{(2\sqrt{5}+\sqrt{2})(\sqrt{5}+3\sqrt{2})}{(\sqrt{5}-3\sqrt{2})(\sqrt{5}+3\sqrt{2})} = \dfrac{10+7\sqrt{10}+6}{5-18}$
$= -\dfrac{16+7\sqrt{10}}{13}.$

8. $\dfrac{a-\sqrt{x}}{a+\sqrt{x}}=\dfrac{(a-\sqrt{x})^2}{(a+\sqrt{x})(a-\sqrt{x})}=\dfrac{a^2-2a\sqrt{x}+x}{a^2-x}.$

9. $\dfrac{\sqrt{a+1}-2}{\sqrt{a+1}-1}=\dfrac{(\sqrt{a+1}-2)(\sqrt{a+1}+1)}{(\sqrt{a+1}-1)(\sqrt{a+1}+1)}=\dfrac{a+1-\sqrt{a+1}-2}{a+1-1}$

$=\dfrac{a-1-\sqrt{a+1}}{a}.$

10. $\dfrac{\sqrt{x+2}-\sqrt{x}}{\sqrt{x+2}+\sqrt{x}}=\dfrac{(\sqrt{x+2}-\sqrt{x})^2}{(\sqrt{x+2}+\sqrt{x})(\sqrt{x+2}-\sqrt{x})}$

$=\dfrac{x+2-2\sqrt{x^2+2x}+x}{x+2-x}=\dfrac{2x+2-2\sqrt{x^2+2x}}{2}$

$=x+1-\sqrt{x^2+2x}.$

11. $\dfrac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}=\dfrac{(a-\sqrt{a^2-1})^2}{(a+\sqrt{a^2-1})(a-\sqrt{a^2-1})}$

$=\dfrac{a^2-2a\sqrt{a^2-1}+a^2-1}{a^2-(a^2-1)}=2a^2-1-2a\sqrt{a^2-1}.$

12. $\dfrac{x+\sqrt{x^2-4}}{x-\sqrt{x^2-4}}=\dfrac{(x+\sqrt{x^2-4})^2}{(x-\sqrt{x^2-4})(x+\sqrt{x^2-4})}$

$=\dfrac{x^2+2x\sqrt{x^2-4}+x^2-4}{x^2-(x^2-4)}=\dfrac{2x^2-4+2x\sqrt{x^2-4}}{4}$

$=\dfrac{x^2-2+x\sqrt{x^2-4}}{2}.$

13. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{(\sqrt{a+x}+\sqrt{a-x})^2}{(\sqrt{a+x}-\sqrt{a-x})(\sqrt{a+x}+\sqrt{a-x})}$

$=\dfrac{a+x+2\sqrt{a^2-x^2}+a-x}{a+x-(a-x)}=\dfrac{2a+2\sqrt{a^2-x^2}}{2x}=\dfrac{a+\sqrt{a^2-x^2}}{x}.$

14. $\dfrac{\sqrt{a^2-1}-\sqrt{a^2+1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}=\dfrac{(\sqrt{a^2-1}-\sqrt{a^2+1})^2}{(\sqrt{a^2-1}+\sqrt{a^2+1})(\sqrt{a^2-1}-\sqrt{a^2+1})}$

$=\dfrac{a^2-1-2\sqrt{a^4-1}+a^2+1}{a^2-1-(a^2+1)}=\dfrac{2a^2-2\sqrt{a^4-1}}{-2}=\sqrt{a^4-1}-a^2.$

Art. 245. — Page 203.

2. $$\frac{3}{\sqrt{2}-1}=\frac{3(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$$
$$=\frac{3\sqrt{2}+3}{2-1}=3\sqrt{2}+3$$
$$=3\times 1.4142\cdots+3$$
$$=4.2426\cdots+3$$
$$=7.243\text{, approximately.}$$

$$\begin{array}{r|l|l} & \dot{2}. & 1.4142\cdots \\ & 1 & \\ \hline 24 & 100 & \\ & 96 & \\ \hline 281 & 400 & \\ & 281 & \\ \hline 2824 & 11900 & \\ & 11296 & \\ \hline 28282 & 60400 & \end{array}$$

3. $$\frac{7}{\sqrt[3]{9}}=\frac{7\sqrt[3]{3}}{\sqrt[3]{9}\sqrt[3]{3}}=\frac{7\sqrt[3]{3}}{\sqrt[3]{27}}=\frac{7\sqrt[3]{3}}{3}$$
$$=\frac{7\times 1.4422\cdots}{3}$$
$$=\frac{10.0954\cdots}{3}$$
$$=3.365\text{, approximately.}$$

$$\begin{array}{r|l|l} & \dot{3}. & 1.4422\cdots \\ & 1 & \\ \hline 300 & 2000 & \\ 120 & & \\ 16 & & \\ \hline 436 & 1744 & \\ \hline 120 & 256000 & \\ 32 & & \\ \hline 58800 & & \\ 1680 & & \\ 16 & & \\ \hline 60496 & 241984 & \\ \hline 1680 & 14016000 & \\ 32 & & \\ \hline 6220800 & & \\ 8640 & & \\ 4 & & \\ \hline 6229444 & 12458888 & \\ \hline 8640 & 1557112000 & \\ 8 & & \\ \hline 623809200 & & \end{array}$$

4. $$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{(\sqrt{3}-\sqrt{2})^2}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$
$$=\frac{3-2\sqrt{6}+2}{3-2}=5-2\sqrt{6}$$
$$=5-2\times 2.4494\cdots$$
$$=5-4.8988\cdots$$
$$=.101\text{, approximately.}$$

$$\begin{array}{r|l|l} & \dot{6}. & 2.4494\cdots \\ & 4 & \\ \hline 44 & 200 & \\ & 176 & \\ \hline 484 & 2400 & \\ & 1936 & \\ \hline 4889 & 46400 & \\ & 44001 & \\ \hline 48984 & 239900 & \end{array}$$

5. $\dfrac{2\sqrt{5}-\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}=\dfrac{(2\sqrt{5}-\sqrt{3})(3\sqrt{5}-2\sqrt{3})}{(3\sqrt{5}+2\sqrt{3})(3\sqrt{5}-2\sqrt{3})}$

$=\dfrac{30-7\sqrt{15}+6}{45-12}=\dfrac{36-7\sqrt{15}}{33}$

$=\dfrac{36-27.1103\cdots}{33}$

$=.269$, approximately.

$$\begin{array}{l|l}
 & 15.\ \ \vert\ 3.8729\cdots \\
 & 9 \\ \hline
68 & 600 \\
 & 544 \\ \hline
767 & 5600 \\
 & 5369 \\ \hline
7742 & 23100 \\
 & 15484 \\ \hline
77449 & 761600
\end{array}$$

Art. 249.—Page 205.

3. $4\sqrt{-3}\times2\sqrt{-2}=4\sqrt{3}\sqrt{-1}\times2\sqrt{2}\sqrt{-1}=8\sqrt{6}\,(\sqrt{-1})^2=-8\sqrt{6}.$

4. $\sqrt{-a^2}\times\sqrt{-x^2}=a\sqrt{-1}\times x\sqrt{-1}=ax\,(\sqrt{-1})^2=-ax.$

5. $-3\sqrt{-a}\times4\sqrt{-b}=-3\sqrt{a}\sqrt{-1}\times4\sqrt{b}\sqrt{-1}$

$=-12\sqrt{ab}(\sqrt{-1})^2=12\sqrt{ab}.$

6. $\sqrt{-3}\times\sqrt{-4}\times\sqrt{-5}=\sqrt{3}\sqrt{-1}\times2\sqrt{-1}\times\sqrt{5}\sqrt{-1}$

$=2\sqrt{15}\,(\sqrt{-1})^3=-2\sqrt{15}\sqrt{-1}=-2\sqrt{-15}.$

7. $(1-2\sqrt{-1})(3+\sqrt{-1})=3-5\sqrt{-1}-2(\sqrt{-1})^2$

$=3-5\sqrt{-1}+2=5-5\sqrt{-1}.$

8. $(4+\sqrt{-7})(8-2\sqrt{-7})=(4+\sqrt{7}\sqrt{-1})\times2(4-\sqrt{7}\sqrt{-1})$

$=2[16-7(\sqrt{-1})^2]=2(16+7)=46.$

9. $(2\sqrt{-3}-3\sqrt{-2})(4\sqrt{-3}+6\sqrt{-2})$

$=(2\sqrt{3}\sqrt{-1}-3\sqrt{2}\sqrt{-1})\times2(2\sqrt{3}\sqrt{-1}+3\sqrt{2}\sqrt{-1})$

$=2[12(\sqrt{-1})^2-18\,(\sqrt{-1})^2]=2(-12+18)=12.$

10. $\sqrt{-1}\times\sqrt{-9}\times\sqrt{-16}\times\sqrt{-25}$

$=\sqrt{-1}\times3\sqrt{-1}\times4\sqrt{-1}\times5\sqrt{-1}=60(\sqrt{-1})^4=60.$

11. $(2-\sqrt{-3})^2=4-4\sqrt{-3}+(\sqrt{-3})^2=4-4\sqrt{-3}-3=1-4\sqrt{-3}.$

12. $(\sqrt{-3}+2\sqrt{-2})^2=(\sqrt{3}\sqrt{-1}+2\sqrt{2}\sqrt{-1})^2$

$=-3-4\sqrt{6}-8=-11-4\sqrt{6}.$

13. $(1+\sqrt{-1})(1-\sqrt{-1})=1-(\sqrt{-1})^2=1-(-1)=1+1=2.$

14. $(a+\sqrt{-b})(a-\sqrt{-b})=a^2-(\sqrt{-b})^2=a^2-(-b)=a^2+b.$

15. $(x\sqrt{-x}+y\sqrt{-y})(x\sqrt{-x}-y\sqrt{-y}) = x^2(\sqrt{-x})^2 - y^2(\sqrt{-y})^2$
$= x^2(-x) - y^2(-y) = y^3 - x^3.$

16. $(1+\sqrt{-1})^2 + (1-\sqrt{-1})^2$
$= 1 + 2\sqrt{-1} + (\sqrt{-1})^2 + 1 - 2\sqrt{-1} + (\sqrt{-1})^2$
$= 1 - 1 + 1 - 1 = 0.$

18. $\dfrac{\sqrt{-6}}{\sqrt{-2}} = \sqrt{\dfrac{-6}{-2}} = \sqrt{3}.$ **19.** $\dfrac{\sqrt{-24}}{\sqrt{-3}} = \sqrt{\dfrac{-24}{-3}} = \sqrt{8} = 2\sqrt{2}.$

20. $\dfrac{\sqrt[4]{-12}}{\sqrt[4]{-3}} = \sqrt[4]{4} = (2^2)^{\frac{1}{4}} = 2^{\frac{2}{4}} = 2^{\frac{1}{2}} = \sqrt{2}.$

21. $\dfrac{\sqrt[6]{-54}}{\sqrt[6]{-2}} = \sqrt[6]{27} = (3^3)^{\frac{1}{6}} = 3^{\frac{3}{6}} = 3^{\frac{1}{2}} = \sqrt{3}.$

Art. 256. — Page 208.

1. $\sqrt{12+2\sqrt{35}} = \sqrt{7+2\sqrt{35}+5} = \sqrt{7} + \sqrt{5}.$

2. $\sqrt{7-2\sqrt{12}} = \sqrt{4-2\sqrt{12}+3} = \sqrt{4} - \sqrt{3} = 2 - \sqrt{3}.$

3. $\sqrt{9+2\sqrt{8}} = \sqrt{8+2\sqrt{8}+1} = \sqrt{8} + \sqrt{1} = 2\sqrt{2} + 1.$

4. $\sqrt{9-4\sqrt{5}} = \sqrt{9-2\sqrt{20}} = \sqrt{5-2\sqrt{20}+4} = \sqrt{5} - \sqrt{4} = \sqrt{5} - 2$

5. $\sqrt{16+6\sqrt{7}} = \sqrt{16+2\sqrt{9\times 7}} = \sqrt{9+2\sqrt{63}+7}$
$= \sqrt{9} + \sqrt{7} = 3 + \sqrt{7}.$

6. $\sqrt{8-\sqrt{60}} = \sqrt{8-2\sqrt{15}} = \sqrt{5-2\sqrt{15}+3} = \sqrt{5} - \sqrt{3}.$

7. $\sqrt{15+4\sqrt{14}} = \sqrt{15+2\sqrt{4\times 14}} = \sqrt{8+2\sqrt{56}+7}$
$= \sqrt{8} + \sqrt{7} = 2\sqrt{2} + \sqrt{7}.$

8. $\sqrt{12-\sqrt{108}} = \sqrt{12-2\sqrt{27}} = \sqrt{9-2\sqrt{27}+3} = \sqrt{9} - \sqrt{3} = 3 - \sqrt{3}$

9. $\sqrt{20-5\sqrt{12}} = \sqrt{20-\sqrt{300}} = \sqrt{20-2\sqrt{75}} = \sqrt{15-2\sqrt{75}+5}$
$= \sqrt{15} - \sqrt{5}.$

10. $\sqrt{14+3\sqrt{20}} = \sqrt{14+\sqrt{180}} = \sqrt{14+2\sqrt{45}} = \sqrt{9+2\sqrt{45}+5}$
$= \sqrt{9} + \sqrt{5} = 3 + \sqrt{5}.$

11. $\sqrt{23+\sqrt{360}} = \sqrt{23+2\sqrt{90}} = \sqrt{18+2\sqrt{90}+5}$
$= \sqrt{18} + \sqrt{5} = 3\sqrt{2} + \sqrt{5}.$

12. $\sqrt{24-2\sqrt{63}}=\sqrt{21-2\sqrt{63}+3}=\sqrt{21}-\sqrt{3}.$

13. $\sqrt{33+20\sqrt{2}}=\sqrt{33+2\sqrt{100\times 2}}=\sqrt{33+2\sqrt{200}}$
$=\sqrt{25+2\sqrt{200}+8}=\sqrt{25}+\sqrt{8}=5+2\sqrt{2}.$

14. $\sqrt{47-6\sqrt{10}}=\sqrt{47-2\sqrt{90}}=\sqrt{45-2\sqrt{90}+2}=\sqrt{45}-\sqrt{2}$
$=3\sqrt{5}-\sqrt{2}.$

15. $\sqrt{67-7\sqrt{72}}=\sqrt{67-\sqrt{49\times 72}}=\sqrt{67-2\sqrt{49\times 18}}$
$=\sqrt{49-2\sqrt{49\times 18}+18}=\sqrt{49}-\sqrt{18}=7-3\sqrt{2}.$

16. $\sqrt{2m-2\sqrt{m^2-n^2}}=\sqrt{m+n-2\sqrt{(m+n)(m-n)}+m-n}$
$=\sqrt{m+n}-\sqrt{m-n}.$

17. $\sqrt{2a+x+2\sqrt{a^2+ax}}=\sqrt{a+x+2\sqrt{(a+x)a}+a}=\sqrt{a+x}+\sqrt{a}.$

Art. 257. — Pages 209, 210.

3. $$\begin{aligned}\sqrt{5x-1}-2&=1\\ \sqrt{5x-1}&=3\\ 5x-1&=9\\ 5x&=10\\ x&=2\end{aligned}$$

4. $$\begin{aligned}5&=\sqrt[3]{2x}+3\\ 2&=\sqrt[3]{2x}\\ 2x&=8\\ x&=4\end{aligned}$$

5. $$\begin{aligned}\sqrt[3]{4x+3}&=3\\ 4x+3&=27\\ 4x&=24\\ x&=6\end{aligned}$$

6. $$\begin{aligned}\sqrt{4x^2-19}-2x&=-1\\ \sqrt{4x^2-19}&=2x-1\\ 4x^2-19&=4x^2-4x+1\\ 4x&=20\\ x&=5\end{aligned}$$

7. $$\begin{aligned}\sqrt{x^2-3x+6}&=2-x\\ x^2-3x+6&=4-4x+x^2\\ x&=-2\end{aligned}$$

8. $$\begin{aligned}\sqrt[3]{x^3-6x^2}-x+2&=0\\ \sqrt[3]{x^3-6x^2}&=x-2\\ x^3-6x^2&=x^3-6x^2+12x-8\\ 12x&=8\\ x&=\frac{2}{3}\end{aligned}$$

9. $\sqrt{x}+\sqrt{x+5}=5$

$$\begin{aligned}\sqrt{x+5}&=5-\sqrt{x}\\ x+5&=25-10\sqrt{x}+x\\ 10\sqrt{x}&=20\\ \sqrt{x}&=2\\ x&=4\end{aligned}$$

10. $\sqrt{x-32}+\sqrt{x}=16$

$$\begin{aligned}\sqrt{x-32}&=16-\sqrt{x}\\ x-32&=256-32\sqrt{x}+x\\ 32\sqrt{x}&=288\\ \sqrt{x}&=9\\ x&=81\end{aligned}$$

11.

$$\begin{aligned}\sqrt{x-3}-\sqrt{x+12}&=-3\\ \sqrt{x-3}+3&=\sqrt{x+12}\\ x-3+6\sqrt{x-3}+9&=x+12\\ 6\sqrt{x-3}&=6\\ \sqrt{x-3}&=1\\ x-3&=1\\ x&=4\end{aligned}$$

12.

$$\begin{aligned}\sqrt{2x-7}+\sqrt{2x+9}&=8\\ \sqrt{2x+9}&=8-\sqrt{2x-7}\\ 2x+9&=64-16\sqrt{2x-7}+2x-7\\ 16\sqrt{2x-7}&=48\\ \sqrt{2x-7}&=3\\ 2x-7&=9\\ x&=8\end{aligned}$$

13.

$$\begin{aligned}\sqrt{3x+10}-\sqrt{3x+25}&=-3\\ \sqrt{3x+10}+3&=\sqrt{3x+25}\\ 3x+10+6\sqrt{3x+10}+9&=3x+25\\ 6\sqrt{3x+10}&=6\\ \sqrt{3x+10}&=1\\ 3x+10&=1\\ x&=-3\end{aligned}$$

14.

$$\begin{aligned}\sqrt{(x-a)^2+2ab+b^2}&=x-a+b\\ x^2-2ax+a^2+2ab+b^2&=x^2+a^2+b^2-2ax+2bx-2ab\\ 2bx&=4ab\\ x&=2a\end{aligned}$$

15. $\sqrt{x^2-3x+5}-\sqrt{x^2-5x-2}=1$

$$\begin{aligned}\sqrt{x^2-3x+5}-1&=\sqrt{x^2-5x-2}\\ x^2-3x+5-2\sqrt{x^2-3x+5}+1&=x^2-5x-2\\ -2\sqrt{x^2-3x+5}&=-2x-8\\ \sqrt{x^2-3x+5}&=x+4\\ x^2-3x+5&=x^2+8x+16\\ -11x&=11\\ x&=-1\end{aligned}$$

16.
$$\sqrt{x}-\sqrt{x-3}=\frac{2}{\sqrt{x}}$$

Clearing of fractions,

$$\begin{aligned} x-\sqrt{x^2-3x}&=2\\ x-2&=\sqrt{x^2-3x}\\ x^2-4x+4&=x^2-3x\\ x&=4 \end{aligned}$$

17.
$$\begin{aligned} \sqrt{x-1}+\sqrt{x+4}&=\sqrt{4x+5}\\ x-1+2\sqrt{x^2+3x-4}+x+4&=4x+5\\ 2\sqrt{x^2+3x-4}&=2x+2\\ \sqrt{x^2+3x-4}&=x+1\\ x^2+3x-4&=x^2+2x+1\\ x&=5 \end{aligned}$$

18.
$$\begin{aligned} \sqrt{x^2+4x+12}+\sqrt{x^2-12x-20}&=8\\ 8-\sqrt{x^2+4x+12}&=\sqrt{x^2-12x-20}\\ 64-16\sqrt{x^2+4x+12}+x^2+4x+12&=x^2-12x-20\\ -16\sqrt{x^2+4x+12}&=-16x-96\\ \sqrt{x^2+4x+12}&=x+6\\ x^2+4x+12&=x^2+12x+36\\ -8x&=24\\ x&=-3 \end{aligned}$$

19.
$$\begin{aligned} \frac{\sqrt{x}-3}{\sqrt{x}+7}&=\frac{\sqrt{x}-4}{\sqrt{x}+1}\\ x-2\sqrt{x}-3&=x+3\sqrt{x}-28\\ -5\sqrt{x}&=-25\\ \sqrt{x}&=5\\ x&=25 \end{aligned}$$

20.
$$\sqrt{3x}+\sqrt{3x+13}=\frac{91}{\sqrt{3x+13}}$$

Clearing of fractions,

$$\begin{aligned} \sqrt{9x^2+39x}+3x+13&=91\\ \sqrt{9x^2+39x}&=78-3x\\ 9x^2+39x&=6084-468x+9x^2\\ 507x&=6084\\ x&=12 \end{aligned}$$

21. $$\sqrt{x+1}+\sqrt{x-2}-\sqrt{4x-3}=0$$
$$\sqrt{x+1}+\sqrt{x-2}=\sqrt{4x-3}$$
$$x+1+2\sqrt{x^2-x-2}+x-2=4x-3$$
$$2\sqrt{x^2-x-2}=2x-2$$
$$\sqrt{x^2-x-2}=x-1$$
$$x^2-x-2=x^2-2x+1$$
$$x=3$$

22. $$\sqrt{x}+\sqrt{x+a}=\frac{2a}{\sqrt{x+a}}$$

Clearing of fractions,
$$\sqrt{x^2+ax}+x+a=2a$$
$$\sqrt{x^2+ax}=a-x$$
$$x^2+ax=a^2-2ax+x^2$$
$$3ax=a^2$$
$$x=\frac{a}{3}$$

23. $$\sqrt{\{9+x\sqrt{x^2-3}\}}=x-3$$
$$9+x\sqrt{x^2-3}=x^2-6x+9$$
$$x\sqrt{x^2-3}=x^2-6x$$
$$\sqrt{x^2-3}=x-6$$
$$x^2-3=x^2-12x+36$$
$$12x=39$$
$$x=\frac{13}{4}$$

24. $$\frac{a}{\sqrt{a-x}}-\frac{x}{\sqrt{b-x}}=\sqrt{b-x}.$$

Clearing of fractions,
$$a\sqrt{b-x}-x\sqrt{a-x}=(b-x)\sqrt{a-x}$$
$$=b\sqrt{a-x}-x\sqrt{a-x}$$
$$a\sqrt{b-x}=b\sqrt{a-x}$$
$$a^2b-a^2x=b^2a-b^2x$$
$$a^2x-b^2x=a^2b-b^2a$$
$$x(a+b)(a-b)=ab(a-b)$$
$$x=\frac{ab}{a+b}$$

25.
$$\sqrt{x+a}+\sqrt{x+b}=\sqrt{4x+a+3b}$$
$$x+a+2\sqrt{(x+a)(x+b)}+x+b=4x+a+3b$$
$$2\sqrt{(x+a)(x+b)}=2x+2b$$
$$\sqrt{(x+a)(x+b)}=x+b$$
$$x^2+ax+bx+ab=x^2+2bx+b^2$$
$$ax-bx=b^2-ab$$
$$x(a-b)=-b(a-b)$$
$$x=-b$$

26.
$$\sqrt{\{1+x\sqrt{x^2+16}\}}=x+1$$
$$1+x\sqrt{x^2+16}=x^2+2x+1$$
$$x\sqrt{x^2+16}=x^2+2x$$
$$\sqrt{x^2+16}=x+2$$
$$x^2+16=x^2+4x+4$$
$$4x=12$$
$$x=3$$

27.
$$\sqrt{\{a^2-2ax+x^2\sqrt{3a-x}\}}=a-x$$
$$a^2-2ax+x^2\sqrt{3a-x}=a^2-2ax+x^2$$
$$x^2\sqrt{3a-x}=x^2$$
$$\sqrt{3a-x}=1$$
$$3a-x=1$$
$$x=3a-1$$

CHAPTER XXI.

Art. 259. — Page 212.

3.
$$4x^2 - 7 = 29$$
$$4x^2 = 36$$
$$x^2 = 9$$
$$x = \pm 3$$

4.
$$5x^2 + 5 = 3x^2 + 55$$
$$2x^2 = 50$$
$$x^2 = 25$$
$$x = \pm 5$$

5.
$$\frac{5}{6x^2} - \frac{7}{4x^2} = -\frac{33}{16}$$
Multiplying by $48x^2$,
$$40 - 84 = -99x^2$$
$$99x^2 = 44$$
$$x^2 = \frac{4}{9}$$
$$x = \pm\frac{2}{3}$$

6.
$$4 - \sqrt{3x^2 + 16} = 6$$
$$-\sqrt{3x^2 + 16} = 2$$
$$3x^2 + 16 = 4$$
$$3x^2 = -12$$
$$x^2 = -4$$
$$x = \pm\sqrt{-4} = \pm 2\sqrt{-1}$$

7.
$$ax^2 + b = c$$
$$ax^2 = c - b$$
$$x^2 = \frac{c - b}{a}$$
$$x = \pm\sqrt{\frac{c - b}{a}}$$

8.
$$\frac{5}{4 - x} = \frac{8}{3} - \frac{5}{4 + x}$$
Clearing of fractions,
$$15(4 + x) = 8(16 - x^2) - 15(4 - x)$$
$$60 + 15x = 128 - 8x^2 - 60 + 15x$$
$$8x^2 = 8$$
$$x^2 = 1$$
$$x = \pm 1$$

9.
$$2(x + 3)(x - 3) = (x + 1)^2 - 2x$$
$$2x^2 - 18 = x^2 + 2x + 1 - 2x$$
$$x^2 = 19$$
$$x = \pm\sqrt{19}$$

10.
$$(3x - 2)(2x + 5) + (5x + 1)(4x - 3) - 91 = 0$$
$$6x^2 + 11x - 10 + 20x^2 - 11x - 3 - 91 = 0$$
$$26x^2 = 104$$
$$x^2 = 4$$
$$x = \pm 2$$

11. $$\frac{x^2}{2} - 3 + \frac{5x^2}{12} = \frac{7}{24} - x^2 + \frac{335}{24}$$

Multiplying by 24, $$12x^2 - 72 + 10x^2 = 7 - 24x^2 + 335$$
$$46x^2 = 414$$
$$x^2 = 9$$
$$x = \pm 3$$

12. $$\frac{2x^2-5}{3} - \frac{3x^2+2}{7} - \frac{x^2-10}{6} = 0$$

Multiplying by 42,
$$14(2x^2-5) - 6(3x^2+2) - 7(x^2-10) = 0$$
$$28x^2 - 70 - 18x^2 - 12 - 7x^2 + 70 = 0$$
$$3x^2 = 12$$
$$x^2 = 4$$
$$x = \pm 2$$

13. $$\frac{a}{x^2-b} = \frac{b}{x^2-a}$$

Clearing of fractions,
$$ax^2 - a^2 = bx^2 - b^2$$
$$ax^2 - bx^2 = a^2 - b^2$$
$$x^2(a-b) = (a+b)(a-b)$$
$$x^2 = a+b$$
$$x = \pm\sqrt{a+b}$$

14. $$\frac{4x^2-3}{2x^2-1} = \frac{2(9x^2+2)}{3(3x^2+2)}$$

Clearing of fractions,
$$36x^4 - 3x^2 - 18 = 36x^4 - 10x^2 - 4$$
$$7x^2 = 14$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

15. $$(2x-a)(x-b) + (2x+a)(x+b) = a^2 + b^2$$
$$2x^2 - ax - 2bx + ab + 2x^2 + ax + 2bx + ab = a^2 + b^2$$
$$4x^2 = a^2 - 2ab + b^2$$
$$x^2 = \frac{(a-b)^2}{4}$$
$$x = \pm\frac{a-b}{2}$$

16. $$\frac{5x^2-1}{x^2-3} - \frac{3x^2+1}{x^2+2} - \frac{89}{(x^2-3)(x^2+2)} = 2$$

Multiplying each term by $(x^2-3)(x^2+2)$,
$$(5x^2-1)(x^2+2) - (3x^2+1)(x^2-3) - 89 = 2(x^2-3)(x^2+2)$$
$$5x^4 + 9x^2 - 2 - 3x^4 + 8x^2 + 3 - 89 = 2x^4 - 2x^2 - 12$$
$$19x^2 = 76$$
$$x^2 = 4$$
$$x = \pm 2$$

17. $$x + \sqrt{x^2+3} = \frac{6}{\sqrt{x^2+3}}$$

Clearing of fractions,

$$x\sqrt{x^2+3} + x^2 + 3 = 6$$
$$x\sqrt{x^2+3} = 3 - x^2$$

Squaring,

$$x^4 + 3x^2 = 9 - 6x^2 + x^4$$
$$9x^2 = 9$$
$$x^2 = 1$$
$$x = \pm 1$$

18. $$\frac{1}{1-\sqrt{1-x^2}} - \frac{1}{1+\sqrt{1-x^2}} = \frac{\sqrt{3}}{x^2}$$

Clearing of fractions,

$$x^2(1+\sqrt{1-x^2}) - x^2(1-\sqrt{1-x^2}) = \sqrt{3}\,[1-(1-x^2)]$$
$$x^2 + x^2\sqrt{1-x^2} - x^2 + x^2\sqrt{1-x^2} = \sqrt{3}\,(1-1+x^2)$$
$$2x^2\sqrt{1-x^2} = x^2\sqrt{3}$$
$$2\sqrt{1-x^2} = \sqrt{3}$$
$$4 - 4x^2 = 3$$
$$4x^2 = 1$$
$$x^2 = \frac{1}{4}$$
$$x = \pm\frac{1}{2}$$

Art. 262. — Page 215.

3. $$x^2 + 4x = 5$$

Completing the square,

$$x^2 + 4x + 4 = 9$$

Extracting the square root,

$$x + 2 = \pm 3$$
$$x = -2 \pm 3$$
$$x = 1 \text{ or } -5$$

4. $$x^2 - 5x = -4$$

Completing the square,

$$x^2 - 5x + \frac{25}{4} = \frac{9}{4}$$

Extracting the square root,

$$x - \frac{5}{2} = \pm\frac{3}{2}$$
$$x = \frac{5}{2} \pm \frac{3}{2}$$
$$x = 4 \text{ or } 1$$

5. $$x^2 - 7x = -12$$

Completing the square,

$$x^2 - 7x + \frac{49}{4} = \frac{1}{4}$$

Extracting the square root,

$$x - \frac{7}{2} = \pm\frac{1}{2}$$
$$x = \frac{7}{2} \pm \frac{1}{2}$$
$$x = 4 \text{ or } 3$$

6. $$x^2 + x = 6$$

Completing the square,

$$x^2 + x + \frac{1}{4} = \frac{25}{4}$$

Extracting the square root,

$$x + \frac{1}{2} = \pm\frac{5}{2}$$

$$x = -\frac{1}{2} \pm \frac{5}{2}$$

$$x = 2 \text{ or } -3$$

7. $$3x^2 - 4x = 4$$

$$x^2 - \frac{4x}{3} = \frac{4}{3}$$

Completing the square,

$$x^2 - \frac{4x}{3} + \frac{4}{9} = \frac{16}{9}$$

Extracting the square root,

$$x - \frac{2}{3} = \pm\frac{4}{3}$$

$$x = \frac{2}{3} \pm \frac{4}{3}$$

$$x = 2 \text{ or } -\frac{2}{3}$$

8. $$2x^2 + 5x = -2$$

$$x^2 + \frac{5x}{2} = -1$$

Completing the square,

$$x^2 + \frac{5x}{2} + \frac{25}{16} = \frac{9}{16}$$

Extracting the square root,

$$x + \frac{5}{4} = \pm\frac{3}{4}$$

$$x = -\frac{5}{4} \pm \frac{3}{4}$$

$$x = -2 \text{ or } -\frac{1}{2}$$

9. $$4x^2 - 8x + 3 = 0$$

$$x^2 - 2x = -\frac{3}{4}$$

Completing the square,

$$x^2 - 2x + 1 = \frac{1}{4}$$

Extracting the square root,

$$x - 1 = \pm\frac{1}{2}$$

$$x = 1 \pm \frac{1}{2}$$

$$x = \frac{3}{2} \text{ or } \frac{1}{2}$$

10. $$4x^2 - 3 = 11x$$

$$x^2 - \frac{11x}{4} = \frac{3}{4}$$

Completing the square,

$$x^2 - \frac{11x}{4} + \frac{121}{64} = \frac{169}{64}$$

Extracting the square root,

$$x - \frac{11}{8} = \pm\frac{13}{8}$$

$$x = \frac{11}{8} \pm \frac{13}{8}$$

$$x = 3 \text{ or } -\frac{1}{4}$$

11. $$3 - x - 2x^2 = 0$$

$$-2x^2 - x = -3$$

Dividing by -2,

$$x^2 + \frac{x}{2} = \frac{3}{2}$$

Completing the square,

$$x^2 + \frac{x}{2} + \frac{1}{16} = \frac{25}{16}$$

Extracting the square root,

$$x + \frac{1}{4} = \pm\frac{5}{4}$$

$$x = -\frac{1}{4} \pm \frac{5}{4}$$

$$x = 1 \text{ or } -\frac{3}{2}$$

12. $14 + 15x - 9x^2 = 0$

$$-9x^2 + 15x = -14$$

Dividing by -9,

$$x^2 - \frac{5x}{3} = \frac{14}{9}$$

Completing the square,

$$x^2 - \frac{5x}{3} + \frac{25}{36} = \frac{81}{36}$$

Extracting the square root,

$$x - \frac{5}{6} = \pm\frac{9}{6}$$

$$x = \frac{5}{6} \pm \frac{9}{6}$$

$$x = \frac{7}{3} \text{ or } -\frac{2}{3}$$

Art. 263.—Page 216.

2. $4x^2 + 3x = 10$

Completing the square,

$$4x^2 + 3x + \frac{9}{16} = \frac{169}{16}$$

Extracting the square root,

$$2x + \frac{3}{4} = \pm\frac{13}{4}$$

$$2x = -\frac{3}{4} \pm \frac{13}{4}$$

$$2x = -4 \text{ or } \frac{5}{2}$$

$$x = -2 \text{ or } \frac{5}{4}$$

3. $9x^2 + 2x = 11$

Completing the square,

$$9x^2 + 2x + \frac{1}{9} = \frac{100}{9}$$

Extracting the square root,

$$3x + \frac{1}{3} = \pm\frac{10}{3}$$

$$3x = -\frac{1}{3} \pm \frac{10}{3}$$

$$3x = 3 \text{ or } -\frac{11}{3}$$

$$x = 1 \text{ or } -\frac{11}{9}$$

4. $25x^2 - 15x = -2$

Completing the square,

$$25x^2 - 15x + \frac{9}{4} = \frac{1}{4}$$

Extracting the square root,

$$5x - \frac{3}{2} = \pm\frac{1}{2}$$

$$5x = \frac{3}{2} \pm \frac{1}{2}$$

$$5x = 2 \text{ or } 1$$

$$x = \frac{2}{5} \text{ or } \frac{1}{5}$$

5. $4x^2 - 7x = -3$

Completing the square,

$$4x^2 - 7x + \frac{49}{16} = \frac{1}{16}$$

Extracting the square root,

$$2x - \frac{7}{4} = \pm\frac{1}{4}$$

$$2x = \frac{7}{4} \pm \frac{1}{4}$$

$$2x = 2 \text{ or } \frac{3}{2}$$

$$x = 1 \text{ or } \frac{3}{4}$$

6. $2x^2 + 15x = -13$

Multiplying by 2,

$$4x^2 + 30x = -26$$

Completing the square,

$$4x^2 + 30x + \frac{225}{4} = \frac{121}{4}$$

Extracting the square root,

$$2x+\frac{15}{2}=\pm\frac{11}{2}$$

$$2x=-\frac{15}{2}\pm\frac{11}{2}$$

$$2x=-2 \text{ or } -13$$

$$x=-1 \text{ or } -\frac{13}{2}$$

7. $8x^2+x-34=0$

Multiplying by 2,

$$16x^2+2x=68$$

Completing the square,

$$16x^2+2x+\frac{1}{16}=\frac{1089}{16}$$

Extracting the square root,

$$4x+\frac{1}{4}=\pm\frac{33}{4}$$

$$4x=-\frac{1}{4}\pm\frac{33}{4}$$

$$4x=8 \text{ or } -\frac{17}{2}$$

$$x=2 \text{ or } -\frac{17}{8}$$

8.

$$11x+12-36x^2=0$$

$$36x^2-11x=12$$

Completing the square,

$$36x^2-11x+\frac{121}{144}=\frac{1849}{144}$$

Extracting the square root,

$$6x-\frac{11}{12}=\pm\frac{43}{12}$$

$$6x=\frac{11}{12}\pm\frac{43}{12}$$

$$6x=\frac{9}{2} \text{ or } -\frac{8}{3}$$

$$x=\frac{3}{4} \text{ or } -\frac{4}{9}$$

9. $6x^2-5x=-1$

Multiplying by 6,

$$36x^2-30x=-6$$

Completing the square,

$$36x^2-30x+\frac{25}{4}=\frac{1}{4}$$

Extracting the square root,

$$6x-\frac{5}{2}=\pm\frac{1}{2}$$

$$6x=\frac{5}{2}\pm\frac{1}{2}$$

$$6x=3 \text{ or } 2$$

$$x=\frac{1}{2} \text{ or } \frac{1}{3}$$

10.

$$32x^2+20x-7=0$$

Multiplying by 2,

$$64x^2+40x=14$$

Completing the square,

$$64x^2+40x+\frac{25}{4}=\frac{81}{4}$$

Extracting the square root,

$$8x+\frac{5}{2}=\pm\frac{9}{2}$$

$$8x=-\frac{5}{2}\pm\frac{9}{2}$$

$$8x=2 \text{ or } -7$$

$$x=\frac{1}{4} \text{ or } -\frac{7}{8}$$

11. $48x^2-32x=3$

Multiplying by 3,

$$144x^2-96x=9$$

Completing the square,

$$144x^2-96x+16=25$$

Extracting the square root,

$$12x-4=\pm 5$$

$$12x=4\pm 5$$

$$12x=9 \text{ or } -1$$

$$x=\frac{3}{4} \text{ or } -\frac{1}{12}$$

Art. 265. — Page 218.

3. $2x^2+5x=3$

Multiplying by 8,

$$16x^2+40x=24$$

Completing the square,

$$16x^2+40x+25=49$$

Extracting the square root,

$$4x+5=\pm 7$$
$$4x=-5\pm 7$$
$$4x=-12 \text{ or } 2$$
$$x=-3 \text{ or } \frac{1}{2}$$

4. $4x^2-x=3$

Multiplying by 16,

$$64x^2-16x=48$$

Completing the square,

$$64x^2-16x+1=49$$

Extracting the square root,

$$8x-1=\pm 7$$
$$8x=1\pm 7$$
$$8x=8 \text{ or } -6$$
$$x=1 \text{ or } -\frac{3}{4}$$

5. $x^2-3x=18$

Multiplying by 4,

$$4x^2-12x=72$$

Completing the square,

$$4x^2-12x+9=81$$

Extracting the square root,

$$2x-3=\pm 9$$
$$2x=3\pm 9$$
$$2x=12 \text{ or } -6$$
$$x=6 \text{ or } -3$$

6. $3x^2+4x=4$

Multiplying by 3,

$$9x^2+12x=12$$

Completing the square,

$$9x^2+12x+4=16$$

Extracting the square root,

$$3x+2=\pm 4$$
$$3x=-2\pm 4$$
$$3x=-6 \text{ or } 2$$
$$x=-2 \text{ or } \frac{2}{3}$$

7. $8x^2+2x=3$

Multiplying by 8,

$$64x^2+16x=24$$

Completing the square,

$$64x^2+16x+1=25$$

Extracting the square root,

$$8x+1=\pm 5$$
$$8x=-1\pm 5$$
$$8x=4 \text{ or } -6$$
$$x=\frac{1}{2} \text{ or } -\frac{3}{4}$$

8. $2x^2-7x=15$

Multiplying by 8,

$$16x^2-56x=120$$

Completing the square,

$$16x^2-56x+49=169$$

Extracting the square root,

$$4x-7=\pm 13$$
$$4x=7\pm 13$$
$$4x=20 \text{ or } -6$$
$$x=5 \text{ or } -\frac{3}{2}$$

9. $7x^2-16x+4=0$

Multiplying by 7,

$$49x^2-112x=-28$$

Completing the square,

$$49x^2-112x+64=36$$

Extracting the square root,

$$7x-8=\pm 6$$
$$7x=8\pm 6$$
$$7x=14 \text{ or } 2$$
$$x=2 \text{ or } \frac{2}{7}$$

10. $17x+20=-3x^2$

Multiplying by 12,

$$36x^2+204x=-240$$

Completing the square,

$$36x^2+204x+289=49$$

Extracting the square root,

$$6x+17=\pm 7$$
$$6x=-17\pm 7$$
$$6x=-24 \text{ or } -10$$
$$x=-4 \text{ or } -\frac{5}{3}$$

11. $5x^2-3=14x$

Multiplying by 5,

$$25x^2-70x=15$$

Completing the square,

$$25x^2-70x+49=64$$

Extracting the square root,

$$5x-7=\pm 8$$
$$5x=7\pm 8$$
$$5x=15 \text{ or } -1$$
$$x=3 \text{ or } -\frac{1}{5}$$

12. $2+x-6x^2=0$

Multiplying by -24,

$$144x^2-24x=48$$

Completing the square,

$$144x^2-24x+1=49$$

Extracting the square root,

$$12x-1=\pm 7$$
$$12x=1\pm 7$$
$$12x=8 \text{ or } -6$$
$$x=\frac{2}{3} \text{ or } -\frac{1}{2}$$

13. $8x^2+6x+1=0$

Multiplying by 8,

$$64x^2+48x=-8$$

Completing the square,

$$64x^2+48x+9=1$$

Extracting the square root,

$$8x+3=\pm 1$$
$$8x=-3\pm 1$$
$$8x=-4 \text{ or } -2$$
$$x=-\frac{1}{2} \text{ or } -\frac{1}{4}$$

14. $7x+3=6x^2$

Multiplying by 24,

$$144x^2-168x=72$$

Completing the square,

$$144x^2-168x+49=121$$

Extracting the square root,

$$12x-7=\pm 11$$
$$12x=7\pm 11$$
$$12x=18 \text{ or } -4$$
$$x=\frac{3}{2} \text{ or } -\frac{1}{3}$$

15. $15x^2-8x=-1$

Multiplying by 15,

$$225x^2-120x=-15$$

Completing the square,

$$225x^2-120x+16=1$$

Extracting the square root,

$$15x-4=\pm 1$$
$$15x=4\pm 1$$
$$15x=5 \text{ or } 3$$
$$x=\frac{1}{3} \text{ or } \frac{1}{5}$$

16.

$$41x-14-15x^2=0$$

Multiplying by -60,

$$900x^2-60\times 41x=-840$$

Completing the square,

$$900x^2-60\times 41x+41^2=841$$

Extracting the square root,

$$30x-41=\pm 29$$
$$30x=41\pm 29$$
$$30x=70 \text{ or } 12$$
$$x=\frac{7}{3} \text{ or } \frac{2}{5}$$

Art. 266.—Pages 218-221.

1. $$\frac{x^2}{2}+\frac{x}{3}+\frac{1}{24}=0$$

Multiplying by 24,
$$12x^2+8x=-1$$
Multiplying by 12,
$$144x^2+96x=-12$$
Completing the square,
$$144x^2+96x+16=4$$
Extracting the square root,
$$12x+4=\pm 2$$
$$12x=-6 \text{ or } -2$$
$$x=-\frac{1}{2} \text{ or } -\frac{1}{6}$$

2. $$\frac{2}{x}+\frac{x}{2}=-\frac{5}{2}$$

Multiplying by $2x$,
$$4+x^2=-5x$$
Multiplying by 4,
$$4x^2+20x=-16$$
Completing the square,
$$4x^2+20x+25=9$$
Extracting the square root,
$$2x+5=\pm 3$$
$$2x=-2 \text{ or } -8$$
$$x=-1 \text{ or } -4$$

3. $$\frac{1}{2x}=\frac{7}{6x^2}-\frac{2}{3}$$

Multiplying by $6x^2$,
$$3x=7-4x^2$$
$$4x^2+3x=7$$
Multiplying by 16,
$$64x^2+48x=112$$
Completing the square,
$$64x^2+48x+9=121$$
Extracting the square root,
$$8x+3=\pm 11$$
$$8x=8 \text{ or } -14$$
$$x=1 \text{ or } -\frac{7}{4}$$

4. $$\frac{2}{5}-\frac{5}{2x}=-\frac{15}{4x^2}$$

Multiplying by $20x^2$,
$$8x^2-50x=-75$$
Multiplying by 8,
$$64x^2-400x=-600$$
Completing the square,
$$64x^2-400x+625=25$$
Extracting the square root,
$$8x-25=\pm 5$$
$$8x=20 \text{ or } 30$$
$$x=\frac{5}{2} \text{ or } \frac{15}{4}$$

5. $$(x+5)(x-5)-(11x+1)=0$$
$$x^2-25-11x-1=0$$
$$x^2-11x=26$$
Multiplying by 4, $$4x^2-44x=104$$
Completing the square, $$4x^2-44x+121=225$$
Extracting the square root, $$2x-11=\pm 15$$
$$2x=26 \text{ or } -4$$
$$x=13 \text{ or } -2$$

6. $$4x(18x-1)=(10x-1)^2$$
$$72x^2-4x=100x^2-20x+1$$
$$28x^2-16x=-1$$
Multiplying by 28, $$28^2x^2-28\times 16x=-28$$

Completing the square, $28^2x^2 - 28 \times 16x + 64 = 36$

Extracting the square root, $28x - 8 = \pm 6$

$$28x = 14 \text{ or } 2$$

$$x = \frac{1}{2} \text{ or } \frac{1}{14}$$

7. $$(3x-5)^2 - (x+2)^2 = -5$$

$$9x^2 - 30x + 25 - x^2 - 4x - 4 = -5$$

$$8x^2 - 34x = -26$$

Multiplying by 8, $64x^2 - 272x = -208$

Completing the square, $64x^2 - 272x + 289 = 81$

Extracting the square root, $8x - 17 = \pm 9$

$$8x = 8 \text{ or } 26$$

$$x = 1 \text{ or } \frac{13}{4}$$

8. $$(x+3)^3 - (x-1)^3 = 19$$

$$x^3 + 9x^2 + 27x + 27 - x^3 + 3x^2 - 3x + 1 = 19$$

$$12x^2 + 24x = -9$$

$$4x^2 + 8x = -3$$

Completing the square, $4x^2 + 8x + 4 = 1$

Extracting the square root, $2x + 2 = \pm 1$

$$2x = -3 \text{ or } -1$$

$$x = -\frac{3}{2} \text{ or } -\frac{1}{2}$$

9. $$(x-1)^2 - (3x+8)^2 - (2x+5)^2 = 0$$

$$x^2 - 2x + 1 - 9x^2 - 48x - 64 - 4x^2 - 20x - 25 = 0$$

$$-12x^2 - 70x = 88$$

Multiplying by -12, $144x^2 + 840x = -1056$

Completing the square, $144x^2 + 840x + 1225 = 169$

Extracting the square root, $12x + 35 = \pm 13$

$$12x = -48 \text{ or } -22$$

$$x = -4 \text{ or } -\frac{11}{6}$$

10.

$$\frac{2x+3}{8+x} - \frac{2x+9}{3x+4} = 0$$

$$\frac{2x+3}{8+x} = \frac{2x+9}{3x+4}$$

Clearing of fractions,

$$6x^2 + 17x + 12 = 2x^2 + 25x + 72$$

$$4x^2 - 8x = 60$$

$$x^2 - 2x = 15$$

Completing the square,

$$x^2 - 2x + 1 = 16$$

Extracting the square root,

$$x - 1 = \pm 4$$

$$x = 5 \text{ or } -3$$

11. $\frac{5}{x}-\frac{3x+1}{x^2}=\frac{1}{4}$

Multiplying by $4x^2$,

$20x-12x-4=x^2$

$x^2-8x=-4$

Completing the square,

$x^2-8x+16=12$

Extracting the square root,

$x-4=\pm\sqrt{12}$

$x=4\pm2\sqrt{3}$

12. $4x-\frac{14-x}{x+1}=14$

Clearing of fractions,

$4x^2+4x-14+x=14x+14$

$4x^2-9x=28$

Multiplying by 16,

$64x^2-144x=448$

Completing the square,

$64x^2-144x+81=529$

Extracting the square root,

$8x-9=\pm23$

$8x=32$ or -14

$x=4$ or $-\frac{7}{4}$

13. $\frac{21}{5-x}-\frac{x}{7}=\frac{25}{7}$

Multiplying by $7(5-x)$,

$147-5x+x^2=125-25x$

$x^2+20x=-22$

Completing the square,

$x^2+20x+100=78$

Extracting the square root,

$x+10=\pm\sqrt{78}$

$x=-10\pm\sqrt{78}$

14. $\frac{3x^2}{x-7}-\frac{1-8x}{10}=\frac{x}{5}$

Multiplying by $10(x-7)$, $30x^2+8x^2-57x+7=2x^2-14x$

$36x^2-43x=-7$

Multiplying by 4×36, $4\times36^2\cdot x^2-144\times43x=-1008$

Completing the square, $72^2x^2-144\times43x+43^2=841$

Extracting the square root, $72x-43=\pm29$

$72x=72$ or 14

$x=1$ or $\frac{7}{36}$

15. $\frac{x}{x-1}-\frac{x-1}{x}=\frac{3}{2}$

Clearing of fractions,

$2x^2-2x^2+4x-2=3x^2-3x$

$3x^2-7x=-2$

Multiplying by 12,

$36x^2-84x=-24$

Completing the square,

$36x^2-84x+49=25$

Extracting the square root,

$6x-7=\pm5$

$6x=12$ or 2

$x=2$ or $\frac{1}{3}$

16. $\frac{x}{5-x}-\frac{5-x}{x}=\frac{15}{4}$

Clearing of fractions,

$4x^2-100+40x-4x^2=75x-15x^2$

$15x^2-35x=100$

Multiplying by $\frac{12}{5}$,

$36x^2-84x=240$

Completing the square,

$36x^2-84x+49=289$

Extracting the square root,

$6x-7=\pm17$

$6x=24$ or -10

$x=4$ or $-\frac{5}{3}$

17.. $$\frac{x+1}{x+2}-\frac{x+3}{x+4}=\frac{8}{3}$$

Clearing of fractions,

$$3x^2+15x+12-3x^2-15x-18=8x^2+48x+64$$
$$8x^2+48x=-70$$
$$4x^2+24x=-35$$

Completing the square, $4x^2+24x+36=1$

Extracting the square root, $2x+6=\pm 1$

$$2x=-7 \text{ or } -5$$
$$x=-\frac{7}{2} \text{ or } -\frac{5}{2}$$

18. $$\sqrt{20+x-x^2}=2x-10$$
$$20+x-x^2=4x^2-40x+100$$
$$5x^2-41x=-80$$

Multiplying by 20, $100x^2-820x=-1600$

Completing the square, $100x^2-820x+1681=81$

Extracting the square root, $10x-41=\pm 9$

$$10x=50 \text{ or } 32$$
$$x=5 \text{ or } \frac{16}{5}$$

19. $$2\sqrt{x}+\frac{2}{\sqrt{x}}=5$$

Clearing of fractions, $2x+2=5\sqrt{x}$

$$4x^2+8x+4=25x$$
$$4x^2-17x=-4$$

Multiplying by 16, $64x^2-16\times 17x=-64$

Completing the square, $64x^2-16\times 17x+289=225$

Extracting the square root, $8x-17=\pm 15$

$$8x=32 \text{ or } 2$$
$$x=4 \text{ or } \frac{1}{4}$$

20. $$\frac{2x-1}{x}-\frac{3x}{3x-1}+\frac{1}{2}=0$$

Clearing of fractions,

$$12x^2-10x+2-6x^2+3x^2-x=0$$
$$9x^2-11x=-2$$

Multiplying by 36, $324x^2-396x=-72$

Completing the square, $324x^2-396x+121=49$

Extracting the square root, $18x-11=\pm 7$

$$18x=18 \text{ or } 4$$
$$x=1 \text{ or } \frac{2}{9}$$

21. $$\frac{x^3 - x^2 + 7}{x^2 + 3x - 1} = x + \frac{11}{3}$$

Clearing of fractions, $3x^3 - 3x^2 + 21 = 3x^3 + 9x^2 - 3x + 11x^2 + 33x - 11$

$$23x^2 + 30x = 32$$

Multiplying by 23, $23^2 \cdot x^2 + 23 \times 30x = 736$

Completing the square,

$$23^2 \cdot x^2 + 23 \times 30x + 225 = 961$$

Extracting the square root, $23x + 15 = \pm 31$

$$23x = -46 \text{ or } 16$$

$$x = -2 \text{ or } \frac{16}{23}$$

22. $$\frac{2x^2 + 3x - 5}{3x^2 + 4x - 1} = \frac{2x^2 - x - 1}{3x^2 - 2x + 7}$$

Clearing of fractions,

$$6x^4 + 5x^3 - 7x^2 + 31x - 35 = 6x^4 + 5x^3 - 9x^2 - 3x + 1$$

$$2x^2 + 34x = 36$$

Multiplying by 2, $4x^2 + 68x = 72$

Completing the square,

$$4x^2 + 68x + 289 = 361$$

Extracting the square root, $2x + 17 = \pm 19$

$$2x = 2 \text{ or } -36$$

$$x = 1 \text{ or } -18$$

23. $$\frac{7}{x^2 - 4} - \frac{3}{x + 2} = \frac{22}{5}$$

Multiplying by $5(x^2 - 4)$,

$$35 - 15(x - 2) = 22(x^2 - 4)$$

$$35 - 15x + 30 = 22x^2 - 88$$

$$22x^2 + 15x = 153$$

Multiplying by 4×22, or 88,

$$4 \times 22^2 \cdot x^2 + 88 \times 15x = 13464$$

Completing the square,

$$44^2 \cdot x^2 + 88 \times 15x + 225 = 13689$$

Extracting the square root, $44x + 15 = \pm 117$

$$44x = -132 \text{ or } 102$$

$$x = -3 \text{ or } \frac{51}{22}$$

24. $$\frac{1}{x^2-1}+\frac{1}{3}=\frac{1}{3(x-1)}+\frac{1}{x+1}$$

Multiplying by $3(x^2-1)$, $$3+x^2-1=x+1+3x-3$$
$$x^2-4x+4=0$$
Extracting the square root, $$x-2=0$$
$$x=2$$

25. $$\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}$$

Clearing of fractions,
$$x^3-7x+6+x^3-2x^2-5x+6=2x^3-3x^2-8x+12$$
$$x^2-4x=0$$
Completing the square, $$x^2-4x+4=4$$
Extracting the square root, $$x-2=\pm 2$$
$$x=4 \text{ or } 0$$

26. $$\frac{12+5x}{12-5x}+\frac{2+x}{x}=\frac{1}{1-5x}$$

Clearing of fractions,
$$12x-55x^2-25x^3+24-118x-15x^2+25x^3=12x-5x^2$$
$$65x^2+118x=24$$
Multiplying by 65, $$65^2\cdot x^2+65\times 118x=1560$$
Completing the square,
$$65^2\cdot x^2+65\times 118x+59^2=5041$$
Extracting the square root, $$65x+59=\pm 71$$
$$65x=-130 \text{ or } 12$$
$$x=-2 \text{ or } \frac{12}{65}$$

27. $$\sqrt{4x-3}-\sqrt{x+1}=1$$
$$\sqrt{4x-3}=1+\sqrt{x+1}$$
$$4x-3=1+2\sqrt{x+1}+x+1$$
$$3x-5=2\sqrt{x+1}$$
$$9x^2-30x+25=4x+4$$
$$9x^2-34x=-21$$
Multiplying by 9, $$81x^2-306x=-189$$
Completing the square, $$81x^2-306x+289=100$$
Extracting the square root, $$9x-17=\pm 10$$
$$9x=27 \text{ or } 7$$
$$x=3 \text{ or } \frac{7}{9}$$

28. $$2\sqrt{x}=\sqrt{x+5}+\frac{3}{\sqrt{x+5}}$$

Clearing of fractions, $$2\sqrt{x^2+5x}=x+5+3$$
$$2\sqrt{x^2+5x}=x+8$$
$$4x^2+20x=x^2+16x+64$$
$$3x^2+4x=64$$
Multiplying by 3, $$9x^2+12x=192$$
Completing the square, $$9x^2+12x+4=196$$
Extracting the square root, $$3x+2=\pm 14$$
$$3x=12 \text{ or } -16$$
$$x=4 \text{ or } -\frac{16}{3}$$

29. $$\frac{x+2}{x-1}=\frac{2x+16}{x+5}-\frac{x-2}{x+1}$$

Clearing of fractions, $$x^3+8x^2+17x+10=2x^3+16x^2-2x-16$$
$$-x^3-2x^2+13x-10$$
$$6x^2-6x=36$$
Multiplying by $\frac{2}{3}$, $$4x^2-4x=24$$
Completing the square, $$4x^2-4x+1=25$$
Extracting the square root, $$2x-1=\pm 5$$
$$2x=6 \text{ or } -4$$
$$x=3 \text{ or } -2$$

30. $$\sqrt{3x+1}+\sqrt{2x-1}=\sqrt{9x+4}$$
$$3x+1+2\sqrt{6x^2-x-1}+2x-1=9x+4$$
$$2\sqrt{6x^2-x-1}=4x+4$$
$$\sqrt{6x^2-x-1}=2x+2$$
$$6x^2-x-1=4x^2+8x+4$$
$$2x^2-9x=5$$
Multiplying by 8, $$16x^2-72x=40$$
Completing the square, $$16x^2-72x+81=121$$
Extracting the square root, $$4x-9=\pm 11$$
$$4x=20 \text{ or } -2$$
$$x=5 \text{ or } -\frac{1}{2}$$

33. $$x^2-2ax=(b+a)(b-a)$$
Completing the square, $$x^2-2ax+a^2=b^2-a^2+a^2=b^2$$
Extracting the square root, $$x-a=\pm b$$
$$x=a+b \text{ or } a-b$$

34. $$x^2 - ax + bx = ab$$
$$x^2 - (a - b)x = ab$$

Multiplying by 4, $$4x^2 - 4(a - b)x = 4ab$$

Completing the square,
$$4x^2 - 4(a - b)x + (a - b)^2 = (a - b)^2 + 4ab$$
$$= a^2 + 2ab + b^2$$

Extracting the square root, $$2x - (a - b) = \pm(a + b)$$
$$2x = a - b \pm (a + b)$$
$$2x = 2a \text{ or } -2b$$
$$x = a \text{ or } -b$$

35. $$x^2 - (a + 1)x = -a$$

Multiplying by 4, $$4x^2 - 4(a + 1)x = -4a$$

Completing the square,
$$4x^2 - 4(a + 1)x + (a + 1)^2 = (a + 1)^2 - 4a$$
$$= a^2 - 2a + 1$$

Extracting the square root, $$2x - (a + 1) = \pm(a - 1)$$
$$2x = a + 1 \pm (a - 1)$$
$$2x = 2 \text{ or } 2a$$
$$x = 1 \text{ or } a$$

36. $$x^2 + 2(c + 8)x = -32c$$

Completing the square,
$$x^2 + 2(c + 8)x + (c + 8)^2 = (c + 8)^2 - 32c$$
$$= c^2 - 16c + 64$$

Extracting the square root, $$x + (c + 8) = \pm(c - 8)$$
$$x = -(c + 8) \pm (c - 8)$$
$$x = -16 \text{ or } -2c$$

37. $$x^2 - m^2(1 - m)x = m^5$$

Multiplying by 4, $$4x^2 - 4m^2(1 - m)x = 4m^5$$

Completing the square,
$$4x^2 - 4m^2(1 - m)x + m^4(1 - m)^2 = m^4(1 - m)^2 + 4m^5$$
$$= m^4 + 2m^5 + m^6$$

Extracting the square root, $$2x - m^2(1 - m) = \pm(m^2 + m^3)$$
$$2x = m^2 - m^3 \pm (m^2 + m^3)$$
$$2x = 2m^2 \text{ or } -2m^3$$
$$x = m^2 \text{ or } -m^3$$

38. $$acx^2 - bcx - adx = -bd$$
$$acx^2 - (bc + ad)x = -bd$$
Multiplying by $4ac$,
$$4a^2c^2x^2 - 4ac(bc + ad)x = -4abcd$$
Completing the square,
$$4a^2c^2x^2 - 4ac(bc + ad)x + (bc + ad)^2 = (bc + ad)^2 - 4abcd$$
$$= b^2c^2 - 2abcd + a^2d^2$$
Extracting the square root, $2acx - (bc + ad) = \pm (bc - ad)$
$$2acx = bc + ad \pm (bc - ad)$$
$$2acx = 2bc \text{ or } 2ad$$
$$x = \frac{b}{a} \text{ or } \frac{d}{c}$$

39. $$(x + 2p)^3 = (x + p)^3 + 37p^3$$
$$x^3 + 6x^2p + 12xp^2 + 8p^3 = x^3 + 3x^2p + 3xp^2 + 38p^3$$
$$3px^2 + 9p^2x = 30p^3$$
$$x^2 + 3px = 10p^2$$
Multiplying by 4, $4x^2 + 12px = 40p^2$
Completing the square, $4x^2 + 12px + 9p^2 = 49p^2$
Extracting the square root, $2x + 3p = \pm 7p$
$$2x = 4p \text{ or } -10p$$
$$x = 2p \text{ or } -5p$$

40. $$6x^2 + 9ax + 2bx = -3ab$$
$$6x^2 + (9a + 2b)x = -3ab$$
Multiplying by 24, $144x^2 + 24(9a + 2b)x = -72ab$
Completing the square,
$$144x^2 + 24(9a + 2b)x + (9a + 2b)^2 = (9a + 2b)^2 - 72ab$$
$$= 81a^2 - 36ab + 4b^2$$
Extracting the square root, $12x + (9a + 2b) = \pm (9a - 2b)$
$$12x = -(9a + 2b) \pm (9a - 2b)$$
$$12x = -18a \text{ or } -4b$$
$$x = -\frac{3a}{2} \text{ or } -\frac{b}{3}$$

41. $$\frac{2x(a - x)}{3a - 2x} = \frac{a}{4}$$
Clearing of fractions, $8ax - 8x^2 = 3a^2 - 2ax$
$$8x^2 - 10ax = -3a^2$$
Multiplying by 8, $64x^2 - 80ax = -24a^2$
Completing the square, $64x^2 - 80ax + 25a^2 = a^2$
Extracting the square root, $8x - 5a = \pm a$
$$8x = 6a \text{ or } 4a$$
$$x = \frac{3a}{4} \text{ or } \frac{a}{2}$$

42. $$\frac{x^2}{x+1}=\frac{m^2}{m+1}$$

Clearing of fractions, $$(m+1)x^2=m^2x+m^2$$
$$(m+1)x^2-m^2x=m^2$$

Multiplying by $4(m+1)$,
$$4(m+1)^2x^2-4m^2(m+1)x=4m^2(m+1)$$

Completing the square,
$$4(m+1)^2x^2-4m^2(m+1)x+m^4=m^4+4m^3+4m^2$$

Extracting the square root, $2(m+1)x-m^2=\pm(m^2+2m)$
$$2(m+1)x=m^2\pm(m^2+2m)$$
$$2(m+1)x=2m^2+2m \text{ or } -2m$$
$$x=m \text{ or } -\frac{m}{m+1}$$

43. $$x+\frac{1}{x}=\frac{a}{b}+\frac{b}{a}$$

Clearing of fractions, $$abx^2+ab=a^2x+b^2x$$
$$abx^2-(a^2+b^2)x=-ab$$

Multiplying by $4ab$, $4a^2b^2x^2-4ab(a^2+b^2)x=-4a^2b^2$

Completing the square,
$$4a^2b^2x^2-4ab(a^2+b^2)x+(a^2+b^2)^2=(a^2+b^2)^2-4a^2b^2$$
$$=a^4-2a^2b^2+b^4$$

Extracting the square root, $2abx-(a^2+b^2)=\pm(a^2-b^2)$
$$2abx=a^2+b^2\pm(a^2-b^2)$$
$$2abx=2a^2 \text{ or } 2b^2$$
$$x=\frac{a}{b} \text{ or } \frac{b}{a}$$

44. $$\frac{x+1}{\sqrt{x}}=\frac{a+1}{\sqrt{a}}$$
$$\frac{x^2+2x+1}{x}=\frac{a^2+2a+1}{a}$$

Clearing of fractions, $$ax^2+2ax+a=a^2x+2ax+x$$
$$ax^2-(a^2+1)x=-a$$

Multiplying by $4a$, $4a^2x^2-4a(a^2+1)x=-4a^2$

Completing the square,
$$4a^2x^2-4a(a^2+1)x+(a^2+1)^2=(a^2+1)^2-4a^2$$
$$=a^4-2a^2+1$$

Extracting the square root, $2ax-(a^2+1)=\pm(a^2-1)$
$$2ax=a^2+1\pm(a^2-1)$$
$$2ax=2a^2 \text{ or } 2$$
$$x=a \text{ or } \frac{1}{a}$$

45. $$\sqrt{(a+b)x-4ab}=x-2b$$
$$ax+bx-4ab=x^2-4bx+4b^2$$
$$x^2-ax-5bx=-4ab-4b^2$$
$$x^2-(a+5b)x=-4ab-4b^2$$
Multiplying by 4, $$4x^2-4(a+5b)x=-16ab-16b^2$$
Completing the square,
$$4x^2-4(a+5b)x+(a+5b)^2=(a+5b)^2-16ab-16b^2$$
$$=a^2-6ab+9b^2$$
Extracting the square root, $$2x-(a+5b)=\pm(a-3b)$$
$$2x=a+5b\pm(a-3b)$$
$$2x=2a+2b \text{ or } 8b$$
$$x=a+b \text{ or } 4b$$

46. $$\sqrt{x-4ab}=\frac{(a+b)(a-b)}{\sqrt{x}}$$
Clearing of fractions, $$\sqrt{x^2-4abx}=a^2-b^2$$
$$x^2-4abx=a^4-2a^2b^2+b^4$$
Completing the square, $$x^2-4abx+4a^2b^2=a^4+2a^2b^2+b^4$$
Extracting the square root, $$x-2ab=\pm(a^2+b^2)$$
$$x=2ab\pm(a^2+b^2)$$
$$x=(a+b)^2 \text{ or } -(a-b)^2$$

47. $$2\sqrt{x-m}+3\sqrt{2x}=\frac{7m+5x}{\sqrt{x-m}}$$
Clearing of fractions,
$$2x-2m+3\sqrt{2x^2-2mx}=7m+5x$$
$$3\sqrt{2x^2-2mx}=9m+3x$$
$$\sqrt{2x^2-2mx}=3m+x$$
$$2x^2-2mx=9m^2+6mx+x^2$$
$$x^2-8mx=9m^2$$
Completing the square, $$x^2-8mx+16m^2=25m^2$$
Extracting the square root, $$x-4m=\pm 5m$$
$$x=9m \text{ or } -m$$

48. $$\frac{1}{a+\sqrt{a^2-x}}+\frac{1}{a-\sqrt{a^2-x}}=1+\frac{x}{a}$$
$$\frac{a-\sqrt{a^2-x}+a+\sqrt{a^2-x}}{a^2-(a^2-x)}=\frac{a+x}{a}$$
$$\frac{2a}{x}=\frac{a+x}{a}$$

Clearing of fractions, $2a^2 = ax + x^2$

Multiplying by 4, $4x^2 + 4ax = 8a^2$

Completing the square, $4x^2 + 4ax + a^2 = 9a^2$

Extracting the square root, $2x + a = \pm 3a$

$$2x = 2a \text{ or } -4a$$
$$x = a \text{ or } -2a$$

49. $$\sqrt{x+a} + \sqrt{x+2a} = \sqrt{2x+3a}$$
$$x + a + 2\sqrt{x^2+3ax+2a^2} + x + 2a = 2x + 3a$$
$$2\sqrt{x^2+3ax+2a^2} = 0$$
$$x^2 + 3ax + 2a^2 = 0$$

Multiplying by 4, $4x^2 + 12ax = -8a^2$

Completing the square, $4x^2 + 12ax + 9a^2 = a^2$

Extracting the square root, $2x + 3a = \pm a$

$$2x = -2a \text{ or } -4a$$
$$x = -a \text{ or } -2a$$

50. $$\frac{x^2+1}{x} = \frac{a+b}{c} + \frac{c}{a+b}$$

Clearing of fractions,

$$c(a+b)x^2 + c(a+b) = (a+b)^2x + c^2x$$
$$c(a+b)x^2 - x[(a+b)^2 + c^2] = -c(a+b)$$

Multiplying by $4c(a+b)$,

$$4c^2(a+b)^2x^2 - 4c(a+b)[(a+b)^2+c^2]x = -4c^2(a+b)^2$$

Completing the square,

$$4c^2(a+b)^2x^2 - 4c(a+b)[(a+b)^2 + c^2]x + [(a+b)^2 + c^2]^2$$
$$= [(a+b)^2 + c^2]^2 - 4c^2(a+b)^2$$
$$= (a+b)^4 - 2c^2(a+b)^2 + c^4$$

Extracting the square root,

$$2c(a+b)x - [(a+b)^2 + c^2] = \pm[(a+b)^2 - c^2]$$
$$2c(a+b)x = (a+b)^2 + c^2 \pm [(a+b)^2 - c^2]$$
$$2c(a+b)x = 2(a+b)^2 \text{ or } 2c^2$$
$$x = \frac{a+b}{c} \text{ or } \frac{c}{a+b}$$

51. $$\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}$$

Clearing of fractions,

$$abx = abx + b^2x + bx^2 + a^2x + abx + ax^2 + a^2b + ab^2 + abx$$
$$(a+b)x^2 + (a^2 + 2ab + b^2)x = -a^2b - ab^2$$

Dividing by $a + b$, $x^2 + (a+b)x = -ab$

Multiplying by 4, $4x^2 + 4(a+b)x = -4ab$

Completing the square,

$$4x^2 + 4(a+b)x + (a+b)^2 = (a+b)^2 - 4ab$$
$$= a^2 - 2ab + b^2$$

Extracting the square root,

$$2x + (a+b) = \pm(a-b)$$
$$2x = -(a+b) \pm (a-b)$$
$$2x = -2a \text{ or } -2b$$
$$x = -a \text{ or } -b$$

52. $$x^2 + bx + cx = (a+c)(a-b)$$
$$x^2 + (b+c)x = a^2 - ab + ac - bc$$

Multiplying by 4, $4x^2 + 4(b+c)x = 4a^2 - 4ab + 4ac - 4bc$

Completing the square,

$$4x^2 + 4(b+c)x + (b+c)^2 = (b+c)^2 + 4a^2 - 4ab + 4ac - 4bc$$
$$= 4a^2 - 4ab + 4ac + b^2 - 2bc + c^2$$

Extracting the square root,

$$2x + (b+c) = \pm(2a - b + c)$$
$$2x = -(b+c) \pm (2a - b + c)$$
$$2x = 2a - 2b \text{ or } -2a - 2c$$
$$x = a - b \text{ or } -a - c$$

53. $$abx^2 + \frac{3a^2x}{c} = \frac{6a^2 + ab - 2b^2}{c^2} - \frac{b^2x}{c}$$

Multiplying by c^2, $abc^2x^2 + 3a^2cx = 6a^2 + ab - 2b^2 - b^2cx$

$$abc^2x^2 + c(3a^2 + b^2)x = 6a^2 + ab - 2b^2$$

Multiplying by $4ab$,

$$4a^2b^2c^2x^2 + 4abc(3a^2 + b^2)x = 4ab(6a^2 + ab - 2b^2)$$

Completing the square,

$$4a^2b^2c^2x^2 + 4abc(3a^2 + b^2)x + (3a^2 + b^2)^2 = (3a^2 + b^2)^2 + 24a^3b + 4a^2b^2 - 8ab^3$$
$$= 9a^4 + 24a^3b + 10a^2b^2 - 8ab^3 + b^4$$

Extracting the square root,

$$2abcx + (3a^2 + b^2) = \pm(3a^2 + 4ab - b^2)$$
$$2abcx = -(3a^2 + b^2) \pm (3a^2 + 4ab - b^2)$$
$$2abcx = 4ab - 2b^2 \text{ or } -6a^2 - 4ab$$
$$x = \frac{2a - b}{ac} \text{ or } -\frac{3a + 2b}{bc}$$

54. $$(3a^2 + b^2)(x^2 - x + 1) = (3b^2 + a^2)(x^2 + x + 1)$$
$$(3a^2 + b^2 - 3b^2 - a^2)x^2 + (-3a^2 - b^2 - 3b^2 - a^2)x = 3b^2 + a^2 - 3a^2 - b^2$$
$$2(a^2 - b^2)x^2 - 4(a^2 + b^2)x = -2a^2 + 2b^2$$
$$(a^2 - b^2)x^2 - 2(a^2 + b^2)x = -(a^2 - b^2)$$

Multiplying by $a^2 - b^2$,

$$(a^2 - b^2)^2x^2 - 2(a^4 - b^4)x = -(a^2 - b^2)^2$$

Completing the square,

$$(a^2-b^2)^2x^2-2(a^4-b^4)x+(a^2+b^2)^2=(a^2+b^2)^2-(a^2-b^2)^2$$
$$=4a^2b^2$$

Extracting the square root,

$$(a^2-b^2)x-(a^2+b^2)=\pm 2ab$$
$$(a^2-b^2)x=a^2+b^2\pm 2ab$$
$$(a^2-b^2)x=(a+b)^2 \text{ or } (a-b)^2$$
$$x=\frac{a+b}{a-b} \text{ or } \frac{a-b}{a+b}$$

Art. 267. — Page 222.

3. $a=2,\ b=5,\ c=18.$
$$\therefore x=\frac{-5\pm\sqrt{25+144}}{4}$$
$$=\frac{-5\pm 13}{4}=2 \text{ or } -\frac{9}{2}$$

4. $a=3,\ b=-2,\ c=5.$
$$\therefore x=\frac{2\pm\sqrt{4+60}}{6}$$
$$=\frac{2\pm 8}{6}=\frac{5}{3} \text{ or } -1$$

5. $a=1,\ b=-7,\ c=-10.$
$$\therefore x=\frac{7\pm\sqrt{49-40}}{2}$$
$$=\frac{7\pm 3}{2}=5 \text{ or } 2$$

6. $a=5,\ b=1,\ c=18.$
$$\therefore x=\frac{-1\pm\sqrt{1+360}}{10}$$
$$=\frac{-1\pm 19}{10}=-2 \text{ or } \frac{9}{5}$$

7. $a=6,\ b=7,\ c=-1.$
$$\therefore x=\frac{-7\pm\sqrt{49-24}}{12}$$
$$=\frac{-7\pm 5}{12}=-1 \text{ or } -\frac{1}{6}$$

8. $a=5,\ b=-11,\ c=-2.$
$$\therefore x=\frac{11\pm\sqrt{121-40}}{10}$$
$$=\frac{11\pm 9}{10}=2 \text{ or } \frac{1}{5}$$

9. $4x^2-8x=5$
$a=4,\ b=-8,\ c=5.$
$$\therefore x=\frac{8\pm\sqrt{64+80}}{8}$$
$$=\frac{8\pm 12}{8}=\frac{5}{2} \text{ or } -\frac{1}{2}$$

10. $6x^2+25x=-14$
$a=6,\ b=25,\ c=-14.$
$$\therefore x=\frac{-25\pm\sqrt{625-336}}{12}$$
$$=\frac{-25\pm 17}{12}=-\frac{7}{2} \text{ or } -\frac{2}{3}$$

11. $-9x^2+30x=16$
$a=-9,\ b=30,\ c=16.$
$$\therefore x=\frac{-30\pm\sqrt{900-576}}{-18}$$
$$=\frac{-30\pm 18}{-18}=\frac{8}{3} \text{ or } \frac{2}{3}$$

12. $-10x^2+39x=-27$
$a=-10,\ b=39,\ c=-27.$
$$\therefore x=\frac{-39\pm\sqrt{1521+1080}}{-20}$$
$$=\frac{-39\pm 51}{-20}=\frac{9}{2} \text{ or } -\frac{3}{5}$$

CHAPTER XXII.

Art. 268.—Pages 224–227.

3. Let $x =$ the number of barrels purchased.

Then $\frac{7x}{4} =$ the price in dollars per barrel.

By the conditions, $x \times \frac{7x}{4} = 175$

$$7x^2 = 700$$
$$x^2 = 100$$

Whence, $x = 10$, the number of barrels purchased,

and $\frac{7x}{4} = \frac{70}{4}$ or 17.50, the price in dollars per barrel.

4. Let $x =$ one part.

Then $15 - x =$ the other.

By the conditions, $x^2 + (15 - x)^2 = 117$

$$x^2 + 225 - 30x + x^2 = 117$$
$$2x^2 - 30x = -108$$

Multiplying by 2, $4x^2 - 60x = -216$

Completing the square,

$$4x^2 - 60x + 225 = 9$$
$$2x - 15 = \pm 3$$
$$2x = 18 \text{ or } 12$$

Whence, $x = 9$ or 6,

and $15 - x = 6$ or 9.

Therefore the parts are 9 and 6.

5. Let $x =$ one number.

Then $\frac{126}{x} =$ the other.

By the conditions, $\frac{x}{\frac{126}{x}} = 3\frac{1}{2}$

$$\frac{x^2}{126} = \frac{7}{2}$$
$$x^2 = 441$$

Whence, $x = 21,$

and $\frac{126}{x} = 6.$

Therefore the numbers are 21 and 6.

6. Let $x =$ the no. of hills in the length.

Then $x - 75 =$ the no. in the breadth.

By the conditions, $x(x - 75) = 6250$

$$x^2 - 75x = 6250$$

Multiplying by 4, $4x^2 - 300x = 25000$

Completing the square,

$$4x^2 - 300x + 5625 = 30625$$
$$2x - 75 = \pm 175$$
$$2x = 250 \text{ or } -100$$

Whence, $x = 125$, the no. of hills in the length

and $x - 75 = 50$, the no. in the breadth.

7. Let $x =$ the greater number.

Then $x - 9 =$ the less.

By the conditions, $x(x + x - 9) = 266$

$$2x^2 - 9x = 266$$

Multiplying by 8, $16x^2 - 72x = 2128$

Completing the square,

$$16x^2 - 72x + 81 = 2209$$
$$4x - 9 = \pm 47$$
$$4x = 56 \text{ or } -38$$

Whence, $x = 14$, the greater number,

and $x - 9 = 5$, the less.

8. Let x and $x + 1$ represent the numbers; then by the conditions,

$$x^2 + (x + 1)^2 = 113$$
$$x^2 + x^2 + 2x + 1 = 113$$
$$2x^2 + 2x = 112$$

Multiplying by 2, $4x^2 + 4x = 224$

Completing the square,

$$4x^2 + 4x + 1 = 225$$
$$2x + 1 = \pm 15$$
$$2x = 14 \text{ or } -16$$

Whence, $x = 7,$

and $x + 1 = 8.$

Therefore the numbers are 7 and 8.

9. Let $x =$ the number of cords in one pile.
Then, $26 - x =$ the number of cords in the other.
Also, $x =$ the cost in dimes per cord for the first,
and $26 - x =$ the cost in dimes per cord for the second.
By the conditions,

$$x^2 + (26 - x)^2 = 356$$
$$x^2 + 676 - 52x + x^2 = 356$$
$$2x^2 - 52x = -320$$
$$x^2 - 26x = -160$$

Completing the square,

$$x^2 - 26x + 169 = 9$$
$$x - 13 = \pm 3$$

Whence, $x = 16$ or 10,
and $26 - x = 10$ or 16.
Therefore there were 16 cords in one pile, and 10 in the other.

10. Let $x =$ one number.
Then, $8 - x =$ the other.
By the conditions, $x^3 + (8 - x)^3 = 152$

$$x^3 + 512 - 192x + 24x^2 - x^3 = 152$$
$$24x^2 - 192x = -360$$
$$x^2 - 8x = -15$$

Completing the square, $x^2 - 8x + 16 = 1$

$$x - 4 = \pm 1$$

Whence, $x = 5$ or 3,
and $8 - x = 3$ or 5.
Therefore the numbers are 5 and 3.

11. Let $x - 1$, x, and $x + 1$ represent the numbers; then by the conditions,

$$2(x - 1)(x + 1) = x^2 + 62$$
$$2x^2 - 2 = x^2 + 62$$
$$x^2 = 64$$

Whence, $x = 8$.
Therefore the numbers are 7, 8, and 9.

12. Let $x =$ the number of oxen bought.
Then, $\frac{240}{x} =$ the price paid for each ox.
Since $x - 3$ oxen were sold for \$240 + \$59, or \$299, the price received for each was $\frac{299}{x - 3}$; then by the conditions,

$$\frac{240}{x} = \frac{299}{x - 3} - 8$$

Clearing of fractions, $240x - 720 = 299x - 8x^2 + 24x$

$$8x^2 - 83x = 720$$

Multiplying by 32,

$$256x^2 - 32 \times 83x = 23040$$

Completing the square,

$$256x^2 - 32 \times 83x + 83^2 = 29929$$

$$16x - 83 = \pm 173$$

$$16x = 256 \text{ or } -90$$

Whence, $x = 16$, the number of oxen bought.

13. Let $x =$ the number of barrels bought.

Then, $\frac{96}{x} =$ the cost apiece in dollars.

Also, $\frac{96}{x+8} =$ the cost, if 8 more.

By the conditions, $\frac{96}{x+8} = \frac{96}{x} - 2$

Dividing by 2, $\frac{48}{x+8} = \frac{48}{x} - 1$

Clearing of fractions, $48x = 48x + 384 - x^2 - 8x$

$$x^2 + 8x = 384$$

Completing the square, $x^2 + 8x + 16 = 400$

$$x + 4 = \pm 20$$

Whence, $x = 16$, the number of barrels bought,

and $\frac{96}{x} = 6$, the cost apiece in dollars.

14. Let $x =$ the less number.

Then, $\frac{78}{x} =$ the greater.

By the conditions, $\frac{78}{x^2} = 2 + \frac{1}{x}$

Clearing of fractions, $78 = 2x^2 + x$

Multiplying by 8, $16x^2 + 8x = 624$

Completing the square,

$$16x^2 + 8x + 1 = 625$$

$$4x + 1 = \pm 25$$

$$4x = 24 \text{ or } -26$$

Whence, $x = 6$, the less number,

and $\frac{78}{x} = 13$, the greater.

15. Let $x =$ the width of the frame in inches.

Then $2(18+12)x + 4x^2$,

or $60x + 4x^2 =$ the area of the frame,

and 18×12, or $216 =$ the area of the glass.

By the conditions, $60x + 4x^2 = 216$

Completing the square,

$$4x^2 + 60x + 225 = 441$$
$$2x + 15 = \pm 21$$
$$2x = 6 \text{ or } -36$$

Whence, $x = 3$, the width of the frame in inches.

16. Let $x =$ the cost of the flour in dollars.

Then $x =$ the gain per cent.

By the conditions, $x + x \times \frac{x}{100} = 39$

$$100x + x^2 = 3900$$

Completing the square,

$$x^2 + 100x + 2500 = 6400$$
$$x + 50 = \pm 80$$

Whence, $x = 30$, the cost of the flour in dollars.

17. Let $x =$ the number of persons at first.

Then $\frac{6300}{x} =$ the share of each.

Also, $\frac{6300}{x-2} =$ the amount paid by each survivor.

By the conditions, $\frac{6300}{x-2} = \frac{6300}{x} + 200$

Or, $\frac{63}{x-2} = \frac{63}{x} + 2$

Clearing of fractions, $63x = 63x - 126 + 2x^2 - 4x$

$$2x^2 - 4x = 126$$
$$x^2 - 2x = 63$$

Completing the square,

$$x^2 - 2x + 1 = 64$$
$$x - 1 = \pm 8$$

Whence, $x = 9$, the number of persons at first.

18. Let $x =$ one part,

and $24 - x =$ the other.

By the conditions, $\frac{24}{x} + \frac{24}{24-x} = \frac{64}{15}$

Or, $\frac{3}{x}+\frac{3}{24-x}=\frac{8}{15}$

Clearing of fractions,

$$1080-45x+45x=192x-8x^2$$
$$8x^2-192x=-1080$$
$$x^2-24x=-135$$

Completing the square,

$$x^2-24x+144=9$$
$$x-12=\pm 3$$

Whence, $x=15$ or 9,

and $24-x=9$ or 15.

Therefore the parts are 9 and 15.

19. Let $x=$ the no. of men in front at first.

Then $x+6=$ the no. in depth,

and $x(x+6)$, or $x^2+6x=$ the whole no. of men.

By the conditions, $x^2+6x=4(x+870)$

$$x^2+2x=3480$$

Completing the square,

$$x^2+2x+1=3481$$
$$x+1=\pm 59$$

Whence, $x=58$.

Therefore, $4x+3480=232+3480$

$=3712$, the whole no. of men.

20. Let $x=$ the cost of the goods in dollars.

Then $x=$ the loss per cent.

By the conditions, $x-x\times\frac{x}{100}=16$

$$100x-x^2=1600$$
$$x^2-100x=-1600$$

Completing the square,

$$x^2-100x+2500=900$$
$$x-50=\pm 30$$

Whence, $x=80$ or 20

Therefore the cost of the goods was either \$80 or \$20.

21. Let $x=$ the breadth in rods.

Then $2x=$ the length,

and $x\times 2x$, or $2x^2=$ the area in square rods.

By the conditions,

$$(2x+20)(x+24)=4x^2$$
$$2x^2+68x+480=4x^2$$
$$2x^2-68x=480$$
$$x^2-34x=240$$

Completing the square,

$$x^2-34x+289=529$$
$$x-17=\pm 23$$

Whence, $x=40$

Therefore, $2x^2=3200$, the area in square rods,
$=20$ acres.

22. Let $x=$ the width of the walk in yards.
Then $6x-1=$ the side of the court.
Also, $4x(6x-1)+4x^2$,
or $28x^2-4x=$ the area of the walk in square yards.
By the conditions,

$$28x^2-4x=4(6x-1)+340$$
$$28x^2-28x=336$$

Dividing by 7, $4x^2-4x=48$

Completing the square,

$$4x^2-4x+1=49$$
$$2x-1=\pm 7$$
$$2x=8 \text{ or } -6$$

Whence, $x=4$, the width of the walk in yards,
and $(6x-1)^2=23^2=529$, the area of the court in square yards.

23. Let $2x=$ the number of bushels of barley.
Then $x=$ the cost of the wheat in dimes per bushel,
and $x-4=$ the cost of the barley in dimes per bushel.
By the conditions,

$$54x+2x(x-4)=10(2x+54)+576$$
$$54x+2x^2-8x=20x+540+576$$
$$2x^2+26x=1116$$

Multiplying by 2,

$$4x^2+52x=2232$$

Completing the square,

$$4x^2+52x+169=2401$$
$$2x+13=\pm 49$$

Whence, $2x=36$, the number of bushels of barley,
and $x-4=14$, its cost in dimes per bushel.

24. Let $x =$ the second digit.
Then $2x =$ the first.
Therefore, $20x + x$, or $21x =$ the number,
and
$10x + 2x$, or $12x =$ the number with its digits inverted.
By the conditions,

$$21x(12x + 11) = 4956$$

Dividing by 21, $12x^2 + 11x = 236$
Multiplying by 48,

$$576x^2 + 528x = 11328$$

Completing the square,

$$576x^2 + 528x + 121 = 11449$$
$$24x + 11 = \pm 107$$
$$24x = 96 \text{ or } -118$$

Whence, $x = 4$
Therefore, $21x = 84$, the number required.

25. Let $x =$ the no. of hours taken by the larger.
Then $x + 2 =$ the no. of hours taken by the smaller.
Therefore, $\frac{1}{x} =$ the part filled by the larger in one hour,
and $\frac{1}{x+2} =$ the part filled by the smaller in one hour.
The cistern can be filled by both pipes running together in $2\frac{11}{12}$, or $\frac{35}{12}$ hours; hence both pipes together can fill $\frac{12}{35}$ in one hour. Then by the conditions,

$$\frac{12}{35} = \frac{1}{x} + \frac{1}{x+2}$$

Clearing of fractions,

$$12x^2 + 24x = 35x + 70 + 35x$$
$$12x^2 - 46x = 70$$

Multiplying by 12,

$$144x^2 - 552x = 840$$

Completing the square,

$$144x^2 - 552x + 529 = 1369$$
$$12x - 23 = \pm 37$$
$$12x = 60 \text{ or } -14$$

Whence, $x = 5$, the no. of hours taken by the larger,
and $x + 2 = 7$, the no. of hours taken by the smaller.

26. Let $100x =$ what A put into the firm.

Since A's money was in 12 months, and B's 16 months, A's share of the profits is $\frac{12 \times 100x}{12 \times 100x + 16 \times 3000}$, or $\frac{x}{x+40}$.

Then since the profits were \$1800,

$$100x + \frac{1800x}{x+40} = 2600$$

$$x + \frac{18x}{x+40} = 26$$

Clearing of fractions,

$$x^2 + 40x + 18x = 26x + 1040$$

$$x^2 + 32x = 1040$$

Completing the square,

$$x^2 + 32x + 256 = 1296$$

$$x + 16 = \pm 36$$

Whence, $x = 20$

Therefore, $100x = 2000$, what A puts into the firm.

27. Let $x =$ the rate of the income tax.

Then, $1000 - \frac{x}{100} \times 1000$,

or $1000 - 10x =$ remainder after deducting the income tax.

Also, $1000 - 10x - \frac{x-1}{100}(1000 - 10x)$

$=$ remainder after deducting the percentage.

By the conditions,

$$1000 - 10x - \frac{x-1}{100}(1000 - 10x) = 912$$

$$88 - 10x - 10(x-1) + \frac{x(x-1)}{10} = 0$$

$$98 - 20x + \frac{x(x-1)}{10} = 0$$

Clearing of fractions,

$$980 - 200x + x^2 - x = 0$$

$$x^2 - 201x = -980$$

Multiplying by 4, $4x^2 - 804x = -3920$

Completing the square,

$$4x^2 - 804x + 201^2 = 36481$$

$$2x - 201 = \pm 191$$

$$2x = 10 \text{ or } 392$$

Whence, $x = 5$, the rate of the income tax.

28. Let $x =$ the number of miles an hour.

Then, $x + 3 =$ the number, if three more.

Also, $\frac{102}{x} =$ the time taken to travel 102 miles at x miles an hour,

and $\frac{102}{x+3} =$ the time taken at $x + 3$ miles an hour.

By the conditions, $\frac{102}{x+3} = \frac{102}{x} - \frac{17}{3}$

Dividing by 17, $\frac{6}{x+3} = \frac{6}{x} - \frac{1}{3}$

Clearing of fractions, $18x = 18x + 54 - x^2 - 3x$

$$x^2 + 3x = 54$$

Multiplying by 4,

$$4x^2 + 12x = 216$$

Completing the square,

$$4x^2 + 12x + 9 = 225$$

$$2x + 3 = \pm 15$$

$$2x = 12 \text{ or } -18$$

Whence, $x = 6$, the number of miles an hour.

29. Let $x =$ the length of the edge in inches.

Then, $6x^2 =$ the surface in square inches,

and $12x =$ the sum of the edges.

By the conditions,

$$6x^2 - 12x = 210$$

$$x^2 - 2x = 35$$

Completing the square,

$$x^2 - 2x + 1 = 36$$

$$x - 1 = \pm 6$$

Whence, $x = 7$, the length of the edge in inches.

Therefore, $x^3 = 343$, the volume in cubic inches.

30. Let $x =$ the side of the greater lot in feet.

Then,

$\frac{530 - 4x}{2}$, or $265 - 2x =$ the side of the smaller.

By the conditions,

$$x^2 + (265 - 2x)^2 = 15025$$

$$x^2 + 70225 - 1060x + 4x^2 = 15025$$

$$5x^2 - 1060x = -55200$$

$$x^2 - 212x = -11040$$

Completing the square,

$$x^2 - 212x + 106^2 = 196$$

$$x - 106 = \pm 14$$

Whence, $x = 120$ or 92, the side of the greater lot,

and $265 - 2x = 25$ or 81, the side of the smaller.

Therefore the areas of the lots are 14400 and 625 square feet, or 8464 and 6561 square feet.

31. Let $x =$ the distance CD in miles.

Then, $\frac{x}{19} =$ B's rate in miles per hour.

Therefore, $\frac{x}{19} \times \frac{x}{19}$, or $\frac{x^2}{361} =$ number of miles travelled by B before meeting A.

In $\frac{x}{19}$ hours A travels $\frac{3x}{19}$ miles; he therefore travels in all $28 + \frac{3x}{19}$ miles from C, and is distant $x - 28 - \frac{3x}{19}$ miles from D when he is met by B.

By the conditions, $\frac{x^2}{361} = x - 28 - \frac{3x}{19} = \frac{16x}{19} - 28$

Multiplying by 361,

$$x^2 - 304x = -10108$$

Completing the square,

$$x^2 - 304x + 152^2 = 12996$$

$$x - 152 = \pm 114$$

Whence, $x = 38$ or 266, the distance CD in miles.

32. Let $x =$ the distance PQ in miles.

Then, $\frac{x}{14} =$ the rate of the first courier,

and $\frac{x + 10}{14} =$ the rate of the second.

Therefore, $\frac{20}{\frac{x}{14}}$, or $\frac{280}{x} =$ time taken by the first to travel 20 miles,

and

$\frac{20}{\frac{x+10}{14}}$, or $\frac{280}{x + 10} =$ time taken by the second to travel 20 miles.

By the conditions, $\frac{280}{x + 10} = \frac{280}{x} - \frac{1}{2}$

Clearing of fractions,

$$560x = 560x + 5600 - x^2 - 10x$$
$$x^2 + 10x = 5600$$

Completing the square,

$$x^2 + 10x + 25 = 5625$$
$$x + 5 = \pm 75$$

Whence, $x = 70$, the distance PQ in miles.

33. Let $x =$ the number of shares bought.

Then, $x - 60 =$ the number sold.

Also, $\frac{1500}{x} =$ the price paid per share,

and $\frac{1000}{x-60} =$ the selling price per share

By the conditions, $20 - \frac{1500}{x} = \frac{1000}{x-60} - 20$

Or, $40 - \frac{1500}{x} = \frac{1000}{x-60}$

Dividing by 20, $2 - \frac{75}{x} = \frac{50}{x-60}$

Clearing of fractions,

$$2x^2 - 120x - 75x + 4500 = 50x$$
$$2x^2 - 245x = -4500$$

Multiplying by 8, $16x^2 - 1960x = -36000$

Completing the square,

$$16x^2 - 1960x + 245^2 = 24025$$
$$4x - 245 = \pm 155$$
$$4x = 400 \text{ or } 90$$

Whence, $x = 100$, the number of shares bought,

and $\frac{1500}{x} = 15$, the price paid in dollars.

CHAPTER XXIII.

Art. 270.—Pages 229, 230.

4.

$$x^4 - 25x^2 = -144$$
$$4x^4 - 100x^2 = -576$$
$$4x^4 - 100x^2 + 625 = 49$$
$$2x^2 - 25 = \pm 7$$
$$2x^2 = 18 \text{ or } 32$$
$$x^2 = 9 \text{ or } 16$$
$$x = \pm 3 \text{ or } \pm 4$$

5.

$$x^6 + 20x^3 - 69 = 0$$
$$x^6 + 20x^3 + 100 = 169$$
$$x^3 + 10 = \pm 13$$
$$x^3 = 3 \text{ or } -23$$
$$x = \sqrt[3]{3} \text{ or } -\sqrt[3]{23}$$

6.

$$x^{10} + 31x^5 - 32 = 0$$
$$4x^{10} + 124x^5 = 128$$
$$4x^{10} + 124x^5 + 961 = 1089$$
$$2x^5 + 31 = \pm 33$$
$$2x^5 = 2 \text{ or } -64$$
$$x^5 = 1 \text{ or } -32$$
$$x = 1 \text{ or } -2$$

7.

$$x^{-4} - 9x^{-2} = -20$$
$$4x^{-4} - 36x^{-2} = -80$$
$$4x^{-4} - 36x^{-2} + 81 = 1$$
$$2x^{-2} - 9 = \pm 1$$
$$2x^{-2} = 8 \text{ or } 10$$
$$x^{-2} = 4 \text{ or } 5$$
$$x^{-1} = \pm 2 \text{ or } \pm \sqrt{5}$$
$$x = \pm \frac{1}{2} \text{ or } \pm \frac{1}{\sqrt{5}}$$

$\therefore$ by Art. 243, $\quad x = \pm \frac{1}{2} \text{ or } \pm \frac{1}{5}\sqrt{5}$

8.

$$81x^2 + \frac{1}{x^2} = 82$$
$$81x^4 + 1 = 82x^2$$
$$81^2 \cdot x^4 - 81 \times 82x^2 = -81$$
$$81^2 \cdot x^4 - 81 \times 82x^2 + 41^2 = 1600$$
$$81x^2 - 41 = \pm 40$$
$$81x^2 = 81 \text{ or } 1$$
$$x^2 = 1 \text{ or } \frac{1}{81}$$
$$x = \pm 1 \text{ or } \pm \frac{1}{9}$$

9.
$$8x^6 - 216 = 37x^3$$
$$256x^6 - 32 \times 37x^3 = 6912$$
$$256x^6 - 32 \times 37x^3 + 37^2 = 8281$$
$$16x^3 - 37 = \pm 91$$
$$16x^3 = 128 \text{ or } -54$$
$$x^3 = 8 \text{ or } -\frac{27}{8}$$
$$x = 2 \text{ or } -\frac{3}{2}$$

10.
$$(3x^2 - 2)^2 - 11(3x^2 - 2) + 10 = 0$$
$$9x^4 - 12x^2 + 4 - 33x^2 + 22 + 10 = 0$$
$$9x^4 - 45x^2 = -36$$
$$x^4 - 5x^2 = -4$$
$$4x^4 - 20x^2 = -16$$
$$4x^4 - 20x^2 + 25 = 9$$
$$2x^2 - 5 = \pm 3$$
$$2x^2 = 2 \text{ or } 8$$
$$x^2 = 1 \text{ or } 4$$
$$x = \pm 1 \text{ or } \pm 2$$

11.
$$(x^3 - 5)^2 = 241 - 29x^3$$
$$x^6 - 10x^3 + 25 = 241 - 29x^3$$
$$x^6 + 19x^3 = 216$$
$$4x^6 + 76x^3 = 864$$
$$4x^6 + 76x^3 + 19^2 = 1225$$
$$2x^3 + 19 = \pm 35$$
$$2x^3 = 16 \text{ or } -54$$
$$x^3 = 8 \text{ or } -27$$
$$x = 2 \text{ or } -3$$

12.
$$x^3 - x^{\frac{3}{2}} = 56$$
$$4x^3 - 4x^{\frac{3}{2}} = 224$$
$$4x^3 - 4x^{\frac{3}{2}} + 1 = 225$$
$$2x^{\frac{3}{2}} - 1 = \pm 15$$
$$2x^{\frac{3}{2}} = 16 \text{ or } -14$$
$$x^{\frac{3}{2}} = 8 \text{ or } -7$$
$$x^{\frac{1}{2}} = 2 \text{ or } -\sqrt[3]{7}$$
$$x = 4 \text{ or } \sqrt[3]{49}$$

13.
$$x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$$
$$4x^{\frac{6}{5}} + 4x^{\frac{3}{5}} = 3024$$
$$4x^{\frac{6}{5}} + 4x^{\frac{3}{5}} + 1 = 3025$$
$$2x^{\frac{3}{5}} + 1 = \pm 55$$
$$2x^{\frac{3}{5}} = 54 \text{ or } -56$$
$$x^{\frac{3}{5}} = 27 \text{ or } -28$$
$$x^{\frac{1}{5}} = 3 \text{ or } -\sqrt[3]{28}$$
$$x = 243 \text{ or } -\sqrt[3]{28^5}$$
$$= 243 \text{ or} -28\sqrt[3]{784}$$

14.

$$2x^{\frac{2}{n}} + 3x^{\frac{4}{n}} - 56 = 0$$
$$3x^{\frac{4}{n}} + 2x^{\frac{2}{n}} = 56$$
$$9x^{\frac{4}{n}} + 6x^{\frac{2}{n}} = 168$$
$$9x^{\frac{4}{n}} + 6x^{\frac{2}{n}} + 1 = 169$$
$$3x^{\frac{2}{n}} + 1 = \pm 13$$
$$3x^{\frac{2}{n}} = 12 \text{ or } -14$$
$$x^{\frac{2}{n}} = 4 \text{ or } -\frac{14}{3}$$
$$x^{\frac{1}{n}} = \pm 2 \text{ or } \left(-\frac{14}{3}\right)^{\frac{1}{2}}$$
$$x = (\pm 2)^n \text{ or } \left(-\frac{14}{3}\right)^{\frac{n}{2}}$$

15.

$$3x^{\frac{5}{3}} + x^{\frac{5}{6}} = 3104$$
$$36x^{\frac{5}{3}} + 12x^{\frac{5}{6}} = 37248$$
$$36x^{\frac{5}{3}} + 12x^{\frac{5}{6}} + 1 = 37249$$
$$6x^{\frac{5}{6}} + 1 = \pm 193$$
$$6x^{\frac{5}{6}} = 192 \text{ or } -194$$
$$x^{\frac{5}{6}} = 32 \text{ or } -\frac{97}{3}$$
$$x^{\frac{1}{6}} = 2 \text{ or } -\left(\frac{97}{3}\right)^{\frac{1}{5}}$$
$$x = 64 \text{ or } \left(\frac{97}{3}\right)^{\frac{6}{5}}$$

16.

$$3x^{\frac{3}{2}} + 26x^{\frac{3}{4}} = -16$$
$$9x^{\frac{3}{2}} + 78x^{\frac{3}{4}} = -48$$
$$9x^{\frac{3}{2}} + 78x^{\frac{3}{4}} + 169 = 121$$
$$3x^{\frac{3}{4}} + 13 = \pm 11$$
$$3x^{\frac{3}{4}} = -24 \text{ or } -2$$
$$x^{\frac{3}{4}} = -8 \text{ or } -\frac{2}{3}$$
$$x^{\frac{1}{4}} = -2 \text{ or} -\left(\frac{2}{3}\right)^{\frac{1}{3}}$$
$$x = 16 \text{ or } \left(\frac{2}{3}\right)^{\frac{4}{3}}$$

17.

$$2x^{-5} + 61x^{-\frac{5}{2}} - 96 = 0$$
$$16x^{-5} + 488x^{-\frac{5}{2}} = 768$$
$$16x^{-5} + 488x^{-\frac{5}{2}} + 61^2 = 4489$$
$$4x^{-\frac{5}{2}} + 61 = \pm 67$$
$$4x^{-\frac{5}{2}} = -128 \text{ or } 6$$
$$x^{-\frac{5}{2}} = -32 \text{ or } \frac{3}{2}$$
$$x^{-\frac{1}{2}} = -2 \text{ or } \left(\frac{3}{2}\right)^{\frac{1}{5}}$$
$$x^{-1} = 4 \text{ or } \left(\frac{3}{2}\right)^{\frac{2}{5}}$$
$$x = \frac{1}{4} \text{ or } \left(\frac{2}{3}\right)^{\frac{2}{5}}$$

18.

$$4x - 15 = 17\sqrt{x}$$
$$64x - 16 \times 17\sqrt{x} = 240$$
$$64x - 16 \times 17\sqrt{x} + 17^2 = 529$$
$$8\sqrt{x} - 17 = \pm 23$$
$$8\sqrt{x} = 40 \text{ or } -6$$
$$\sqrt{x} = 5 \text{ or } -\frac{3}{4}$$
$$x = 25 \text{ or } \frac{9}{16}$$

19.
$$\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4}{\sqrt{x}}-1=\frac{4-\sqrt{x}}{\sqrt{x}}$$
$$2x+2\sqrt{x}=16-x$$
$$3x+2\sqrt{x}=16$$
$$9x+6\sqrt{x}+1=48+1=49$$
$$3\sqrt{x}+1=\pm 7$$
$$3\sqrt{x}=6 \text{ or } -8$$
$$\sqrt{x}=2 \text{ or } -\frac{8}{3}$$
$$x=4 \text{ or } \frac{64}{9}$$

20.
$$3x^{\frac{4}{3}}-\frac{5x^{\frac{8}{3}}}{2}=-592$$
$$5x^{\frac{8}{3}}-6x^{\frac{4}{3}}=1184$$
$$25x^{\frac{8}{3}}-30x^{\frac{4}{3}}=5920$$
$$25x^{\frac{8}{3}}-30x^{\frac{4}{3}}+9=5929$$
$$5x^{\frac{4}{3}}-3=\pm 77$$
$$5x^{\frac{4}{3}}=80 \text{ or } -74$$
$$x^{\frac{4}{3}}=16 \text{ or } -\frac{74}{5}$$
$$x^{\frac{1}{3}}=\pm 2 \text{ or } \left(-\frac{74}{5}\right)^{\frac{1}{4}}$$
$$x=\pm 8 \text{ or } \left(-\frac{74}{5}\right)^{\frac{3}{4}}$$

21.
$$8x^{-\frac{6}{5}}-15x^{-\frac{3}{5}}-2=0$$
$$256x^{-\frac{6}{5}}-32\times 15x^{-\frac{3}{5}}=64$$
$$256x^{-\frac{6}{5}}-32\times 15x^{-\frac{3}{5}}+15^2=289$$
$$16x^{-\frac{3}{5}}-15=\pm 17$$
$$16x^{-\frac{3}{5}}=-2 \text{ or } 32$$
$$x^{-\frac{3}{5}}=-\frac{1}{8} \text{ or } 2$$
$$x^{-\frac{1}{5}}=-\frac{1}{2} \text{ or } 2^{\frac{1}{3}}$$
$$x^{-1}=-\frac{1}{32} \text{ or } 2^{\frac{5}{3}}$$
$$x=-32 \text{ or } 2^{-\frac{5}{3}}$$

Art. 271. — Page 232.

4.
$$(x^2-5x)^2-8(x^2-5x)=84$$
$$(x^2-5x)^2-8(x^2-5x)+16=100$$
$$x^2-5x-4=\pm 10$$
$$x^2-5x=14 \text{ or } -6$$
$$4x^2-20x=56 \text{ or } -24$$
$$4x^2-20x+25=81 \text{ or } 1$$
$$2x-5=\pm 9 \text{ or } \pm 1$$
$$2x=14, -4, 6, \text{ or } 4$$
$$x=7, -2, 3, \text{ or } 2$$

5.
$$x^4+10x^3+17x^2-40x-84=0$$
$$x^4+10x^3+25x^2-8x^2-40x=84$$
$$(x^2+5x)^2-8(x^2+5x)=84$$
$$(x^2+5x)^2-8(x^2+5x)+16=100$$
$$x^2+5x-4=\pm 10$$
$$x^2+5x=14 \text{ or } -6$$
$$4x^2+20x=56 \text{ or } -24$$
$$4x^2+20x+25=81 \text{ or } 1$$
$$2x+5=\pm 9 \text{ or } \pm 1$$
$$2x=4, -14, -4, \text{ or } -6$$
$$x=2, -7, -2, \text{ or } -3$$

6.
$$x^2-10x-2\sqrt{x^2-10x+18}+15=0$$
$$x^2-10x+18-2\sqrt{x^2-10x+18}=3$$
$$x^2-10x+18-2\sqrt{x^2-10x+18}+1=4$$
$$\sqrt{x^2-10x+18}-1=\pm 2$$
$$\sqrt{x^2-10x+18}=3 \text{ or } -1$$
$$x^2-10x+18=9 \text{ or } 1$$
$$x^2-10x+25=16 \text{ or } 8$$
$$x-5=\pm 4 \text{ or } \pm\sqrt{8}$$
$$x=1, 9, \text{ or } 5\pm 2\sqrt{2}$$

7.
$$x^2+5+\sqrt{x^2+5}=12$$
$$4(x^2+5)+4\sqrt{x^2+5}=48$$
$$4(x^2+5)+4\sqrt{x^2+5}+1=49$$
$$2\sqrt{x^2+5}+1=\pm 7$$
$$2\sqrt{x^2+5}=6 \text{ or } -8$$
$$\sqrt{x^2+5}=3 \text{ or } -4$$
$$x^2+5=9 \text{ or } 16$$
$$x^2=4 \text{ or } 11$$
$$x=\pm 2 \text{ or } \pm\sqrt{11}$$

8.
$$2x^2+3x-5\sqrt{2x^2+3x+9}=-3$$
$$2x^2+3x+9-5\sqrt{2x^2+3x+9}=6$$
$$4(2x^2+3x+9)-20\sqrt{2x^2+3x+9}+25=49$$
$$2\sqrt{2x^2+3x+9}-5=\pm 7$$
$$2\sqrt{2x^2+3x+9}=12 \text{ or } -2$$
$$\sqrt{2x^2+3x+9}=6 \text{ or } -1$$
$$2x^2+3x+9=36 \text{ or } 1$$
$$2x^2+3x=27 \text{ or } -8$$
$$16x^2+24x=216 \text{ or } -64$$
$$16x^2+24x+9=225 \text{ or } -55$$
$$4x+3=\pm 15 \text{ or } \pm\sqrt{-55}$$
$$4x=12,\ -18, \text{ or } -3\pm\sqrt{-55}$$
$$x=3,\ -\frac{9}{2}, \text{ or } \frac{-3\pm\sqrt{-55}}{4}$$

9.
$$x^4+2x^3-25x^2-26x+120=0$$
$$x^4+2x^3+x^2-26x^2-26x=-120$$
$$(x^2+x)^2-26(x^2+x)=-120$$
$$(x^2+x)^2-26(x^2+x)+169=49$$
$$x^2+x-13=\pm 7$$
$$x^2+x=6 \text{ or } 20$$
$$4x^2+4x=24 \text{ or } 80$$
$$4x^2+4x+1=25 \text{ or } 81$$
$$2x+1=\pm 5 \text{ or } \pm 9$$
$$2x=4,\ -6,\ 8, \text{ or } -10$$
$$x=2,\ -3,\ 4, \text{ or } -5$$

10.
$$x^4-6x^3-29x^2+114x=80$$
$$x^4-6x^3+9x^2-38x^2+114x=80$$
$$(x^2-3x)^2-38(x^2-3x)=80$$
$$(x^2-3x)^2-38(x^2-3x)+361=441$$
$$x^2-3x-19=\pm 21$$
$$x^2-3x=-2 \text{ or } 40$$
$$4x^2-12x=-8 \text{ or } 160$$
$$4x^2-12x+9=1 \text{ or } 169$$
$$2x-3=\pm 1 \text{ or } \pm 13$$
$$2x=2,\ 4,\ -10, \text{ or } 16$$
$$x=1,\ 2,\ -5, \text{ or } 8$$

11.
$$x^2-6x+5\sqrt{x^2-6x+20}=46$$
$$x^2-6x+20+5\sqrt{x^2-6x+20}=66$$
$$4(x^2-6x+20)+20\sqrt{x^2-6x+20}+25=289$$
$$2\sqrt{x^2-6x+20}+5=\pm 17$$
$$2\sqrt{x^2-6x+20}=12 \text{ or } -22$$
$$\sqrt{x^2-6x+20}=6 \text{ or } -11$$
$$x^2-6x+20=36 \text{ or } 121$$
$$x^2-6x+9=25 \text{ or } 110$$
$$x-3=\pm 5 \text{ or } \pm\sqrt{110}$$
$$x=8,\ -2, \text{ or } 3\pm\sqrt{110}$$

12.
$$\sqrt{x+10}-\sqrt[4]{x+10}=2$$
$$4\sqrt{x+10}-4\sqrt[4]{x+10}+1=9$$
$$2\sqrt[4]{x+10}-1=\pm 3$$
$$2\sqrt[4]{x+10}=4 \text{ or } -2$$
$$\sqrt[4]{x+10}=2 \text{ or } -1$$
$$x+10=16 \text{ or } 1$$
$$x=6 \text{ or } -9$$

13.
$$4x^2+6\sqrt{4x^2+12x-2}=-3-12x$$
$$4x^2+12x-2+6\sqrt{4x^2+12x-2}=-5$$
$$4x^2+12x-2+6\sqrt{4x^2+12x-2}+9=4$$
$$\sqrt{4x^2+12x-2}+3=\pm 2$$
$$\sqrt{4x^2+12x-2}=-5 \text{ or } -1$$
$$4x^2+12x-2=25 \text{ or } 1$$
$$4x^2+12x+9=36 \text{ or } 12$$
$$2x+3=\pm 6 \text{ or } \pm\sqrt{12}$$
$$2x=3,\ -9, \text{ or } -3\pm 2\sqrt{3}$$
$$x=\frac{3}{2},\ -\frac{9}{2}, \text{ or } \frac{-3\pm 2\sqrt{3}}{2}$$

14.
$$(x^3+16)^{\frac{2}{3}}-3(x^3+16)^{\frac{1}{3}}+2=0$$
$$4(x^3+16)^{\frac{2}{3}}-12(x^3+16)^{\frac{1}{3}}+9=1$$
$$2(x^3+16)^{\frac{1}{3}}-3=\pm 1$$
$$2(x^3+16)^{\frac{1}{3}}=4 \text{ or } 2$$
$$(x^3+16)^{\frac{1}{3}}=2 \text{ or } 1$$
$$x^3+16=8 \text{ or } 1$$
$$x^3=-8 \text{ or } -15$$
$$x=-2 \text{ or } -\sqrt[3]{15}$$

15.
$$4(x-1)^{\frac{4}{3}}-5(x-1)^{\frac{2}{3}}+1=0$$
$$64(x-1)^{\frac{4}{3}}-80(x-1)^{\frac{2}{3}}+25=9$$
$$8(x-1)^{\frac{2}{3}}-5=\pm 3$$
$$8(x-1)^{\frac{2}{3}}=8 \text{ or } 2$$
$$(x-1)^{\frac{2}{3}}=1 \text{ or } \frac{1}{4}$$
$$(x-1)^{\frac{1}{3}}=\pm 1 \text{ or } \pm\frac{1}{2}$$
$$x-1=\pm 1 \text{ or } \pm\frac{1}{8}$$
$$x=0, 2, \frac{7}{8}, \text{ or } \frac{9}{8}$$

16.
$$x^4+14x^3+47x^2-14x-48=0$$
$$x^4+14x^3+49x^2-2x^2-14x=48$$
$$(x^2+7x)^2-2(x^2+7x)=48$$
$$(x^2+7x)^2-2(x^2+7x)+1=49$$
$$x^2+7x-1=\pm 7$$
$$x^2+7x=8 \text{ or } -6$$
$$4x^2+28x=32 \text{ or } -24$$
$$4x^2+28x+49=81 \text{ or } 25$$
$$2x+7=\pm 9 \text{ or } \pm 5$$
$$2x=2, -16, -2, \text{ or } -12$$
$$x=1, -8, -1, \text{ or } -6$$

17.
$$3(x^2+5x)-2\sqrt{x^2+5x+1}=2$$
$$3(x^2+5x+1)-2\sqrt{x^2+5x+1}=5$$
$$9(x^2+5x+1)-6\sqrt{x^2+5x+1}+1=16$$
$$3\sqrt{x^2+5x+1}-1=\pm 4$$
$$3\sqrt{x^2+5x+1}=-3 \text{ or } 5$$
$$\sqrt{x^2+5x+1}=-1 \text{ or } \frac{5}{3}$$
$$x^2+5x+1=1 \text{ or } \frac{25}{9}$$
$$x^2+5x+\frac{25}{4}=\frac{25}{4} \text{ or } \frac{289}{36}$$
$$x+\frac{5}{2}=\pm\frac{5}{2} \text{ or } \pm\frac{17}{6}$$
$$x=0, -5, \frac{1}{3}, \text{ or } -\frac{16}{3}$$

18. $$(x-a)^{\frac{5}{3}}+2\sqrt{b}\,(x-a)^{\frac{5}{6}}-3b=0$$

$$(x-a)^{\frac{5}{3}}+2\sqrt{b}\,(x-a)^{\frac{5}{6}}+b=4b$$

$$(x-a)^{\frac{5}{6}}+b^{\frac{1}{2}}=\pm 2b^{\frac{1}{2}}$$

$$(x-a)^{\frac{5}{6}}=b^{\frac{1}{2}} \text{ or } -3b^{\frac{1}{2}}$$

$$(x-a)^{\frac{1}{6}}=b^{\frac{1}{10}} \text{ or } -3^{\frac{1}{5}}b^{\frac{1}{10}}$$

$$x-a=b^{\frac{3}{5}} \text{ or } 3\times 3^{\frac{1}{5}}b^{\frac{3}{5}}$$

$$x=a+b^{\frac{3}{5}} \text{ or } a+3\sqrt[5]{3b^3}$$

CHAPTER XXIV.

Art. 274. — Page 234.

2. $\begin{cases} 2x^2 - 3y^2 = -10 & (1) \\ 3x + y = 1 & (2) \end{cases}$

From (2), $y = 1 - 3x$ (3)

Substituting in (1),

$$2x^2 - 3(1 - 3x)^2 = -10$$
$$2x^2 - 3 + 18x - 27x^2 = -10$$
$$25x^2 - 18x = 7$$
$$625x^2 - 450x = 175$$
$$625x^2 - 450x + 81 = 256$$
$$25x - 9 = \pm 16$$
$$25x = 25 \text{ or } -7$$
$$x = 1 \text{ or } -\frac{7}{25}$$

Substituting in (3),

$$y = 1 - 3 \text{ or } 1 + \frac{21}{25}$$
$$= -2 \text{ or } \frac{46}{25}$$

3. $\begin{cases} x + y = -1 & (1) \\ xy = -56 & (2) \end{cases}$

From (1), $y = -x - 1$ (3)

Substituting in (2),

$$x(-x - 1) = -56$$
$$x^2 + x = 56$$
$$4x^2 + 4x = 224$$
$$4x^2 + 4x + 1 = 225$$
$$2x + 1 = \pm 15$$
$$2x = 14 \text{ or } -16$$
$$x = 7 \text{ or } -8$$

Substituting in (3),

$$y = -7 - 1 \text{ or } 8 - 1$$
$$= -8 \text{ or } 7$$

4. $\begin{cases} x - y = 3 & (1) \\ x^2 + y^2 = 117 & (2) \end{cases}$

From (1), $y = x - 3$ (3)

Substituting in (2),

$$x^2 + (x - 3)^2 = 117$$
$$x^2 + x^2 - 6x + 9 = 117$$
$$2x^2 - 6x = 108$$
$$4x^2 - 12x = 216$$
$$4x^2 - 12x + 9 = 225$$
$$2x - 3 = \pm 15$$
$$2x = 18 \text{ or } -12$$
$$x = 9 \text{ or } -6$$

Substituting in (3),

$$y = 6 \text{ or } -9$$

5. $\begin{cases} 10x + y = 3xy & (1) \\ x - y = -2 & (2) \end{cases}$

From (2), $y = x + 2$ (3)

Substituting in (1),

$$10x + x + 2 = 3x(x + 2)$$
$$11x + 2 = 3x^2 + 6x$$
$$3x^2 - 5x = 2$$
$$36x^2 - 60x = 24$$
$$36x^2 - 60x + 25 = 49$$
$$6x - 5 = \pm 7$$
$$6x = 12 \text{ or } -2$$
$$x = 2 \text{ or } -\frac{1}{3}$$

Substituting in (3),

$$y = 2 + 2 \text{ or } -\frac{1}{3} + 2$$
$$= 4 \text{ or } \frac{5}{3}$$

6. $\begin{cases} x^3 - y^3 = -37 & (1) \\ x - y = -1 & (2) \end{cases}$

From (2), $y = x + 1$ (3)

Substituting in (1),

$$x^3 - (x+1)^3 = -37$$
$$-3x^2 - 3x - 1 = -37$$
$$-3x^2 - 3x = -36$$
$$x^2 + x = 12$$
$$4x^2 + 4x = 48$$
$$4x^2 + 4x + 1 = 49$$
$$2x + 1 = \pm 7$$
$$2x = 6 \text{ or } -8$$
$$x = 3 \text{ or } -4$$

Substituting in (3),

$$y = 4 \text{ or } -3$$

7. $\begin{cases} x - y = 5 & (1) \\ xy = -6 & (2) \end{cases}$

From (1), $x = y + 5$ (3)

Substituting in (2),

$$y(y+5) = -6$$
$$y^2 + 5y = -6$$
$$4y^2 + 20y = -24$$
$$4y^2 + 20y + 25 = 1$$
$$2y + 5 = \pm 1$$
$$2y = -6 \text{ or } -4$$
$$y = -3 \text{ or } -2$$

Substituting in (3),

$$x = 2 \text{ or } 3$$

8. $\begin{cases} x + y = 3 & (1) \\ x^2 + y^2 = 29 & (2) \end{cases}$

From (1), $y = 3 - x$ (3)

Substituting in (2),

$$x^2 + (3-x)^2 = 29$$
$$x^2 + 9 - 6x + x^2 = 29$$
$$2x^2 - 6x = 20$$
$$4x^2 - 12x = 40$$
$$4x^2 - 12x + 9 = 49$$
$$2x - 3 = \pm 7$$
$$2x = 10 \text{ or } -4$$
$$x = 5 \text{ or } -2$$

Substituting in (3),

$$y = -2 \text{ or } 5$$

9. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 4 & (1) \\ \dfrac{2}{x} + \dfrac{3}{y} = 1 & (2) \end{cases}$

From (1),

$$3x + 2y = 24 \quad (3)$$
$$2y = 24 - 3x$$
$$y = \frac{24 - 3x}{2} \quad (4)$$

From (2),

$$2y + 3x = xy \quad (5)$$

Comparing (3) and (5),

$$xy = 24 \quad (6)$$

Substituting from (4) in (6),

$$x\left(\frac{24 - 3x}{2}\right) = 24$$
$$24x - 3x^2 = 48$$
$$x^2 - 8x + 16 = 0$$
$$x - 4 = 0$$
$$x = 4$$

Substituting in (4),

$$y = \frac{24 - 12}{2} = 6$$

10. $\begin{cases} x^3 + y^3 = 152 & (1) \\ x + y = 2 & (2) \end{cases}$

From (2), $y = 2 - x$ (3)

Substituting in (1),

$$x^3 + (2-x)^3 = 152$$
$$8 - 12x + 6x^2 = 152$$
$$6x^2 - 12x = 144$$
$$x^2 - 2x = 24$$
$$x^2 - 2x + 1 = 25$$
$$x - 1 = \pm 5$$
$$x = 6 \text{ or } -4$$

Substituting in (3),

$$y = -4 \text{ or } 6$$

11. $\begin{cases} 3x^2 - 2xy = 15 & (1) \\ 2x + 3y = 12 & (2) \end{cases}$

From (2), $3y = 12 - 2x$

$$y = \frac{12 - 2x}{3} \quad (3)$$

Substituting in (1),

$$3x^2 - 2x\left(\frac{12-2x}{3}\right) = 15$$

$$9x^2 - 24x + 4x^2 = 45$$

$$13x^2 - 24x = 45$$

$$169x^2 - 312x = 585$$

$$169x^2 - 312x + 144 = 729$$

$$13x - 12 = \pm 27$$

$$13x = 39 \text{ or } -15$$

$$x = 3 \text{ or } -\frac{15}{13}$$

Substituting in (3),

$$y = \frac{12 - 6}{3} \text{ or } \frac{12 + \frac{30}{13}}{3}$$

$$= 2 \text{ or } \frac{62}{13}$$

12. $\begin{cases} 8x^3 - y^3 = -7 & (1) \\ 2x - y = -1 & (2) \end{cases}$

From (2), $y = 2x + 1 \quad (3)$

Substituting in (1),

$$8x^3 - (2x + 1)^3 = -7$$

$$-12x^2 - 6x - 1 = -7$$

$$-12x^2 - 6x = -6$$

$$2x^2 + x = 1$$

$$16x^2 + 8x = 8$$

$$16x^2 + 8x + 1 = 9$$

$$4x + 1 = \pm 3$$

$$4x = 2 \text{ or } -4$$

$$x = \frac{1}{2} \text{ or } -1$$

Substituting in (3),

$$y = 1 + 1 \text{ or } -2 + 1$$

$$= 2 \text{ or } -1$$

13. $\begin{cases} x^2 + 3xy - y^2 = 23 & (1) \\ x + 2y = 7 & (2) \end{cases}$

From (2), $x = 7 - 2y \quad (3)$

Substituting in (1),

$$(7 - 2y)^2 + 3y(7 - 2y) - y^2 = 23$$

$$49 - 28y + 4y^2 + 21y - 6y^2 - y^2 = 23$$

$$-3y^2 - 7y = -26$$

$$36y^2 + 84y = 312$$

$$36y^2 + 84y + 49 = 361$$

$$6y + 7 = \pm 19$$

$$6y = 12 \text{ or } -26$$

$$y = 2 \text{ or } -\frac{13}{3}$$

Substituting in (3), $x = 7 - 4 \text{ or } 7 + \frac{26}{3}$

$$= 3 \text{ or } \frac{47}{3}$$

14. $\begin{cases} \dfrac{x}{y}+\dfrac{y}{x}=\dfrac{5}{2} & (1) \\ 3x-2y=-4 & (2) \end{cases}$

From (1), $2x^2+2y^2=5xy$ (3)

From (2), $2y=3x+4$

Or, $y=\dfrac{3x+4}{2}$ (4)

Substituting in (3), $2x^2+2\left(\dfrac{3x+4}{2}\right)^2=5x\left(\dfrac{3x+4}{2}\right)$

$$2x^2+\frac{(3x+4)^2}{2}=\frac{15x^2+20x}{2}$$

$$4x^2+9x^2+24x+16=15x^2+20x$$

$$-2x^2+4x=-16$$

$$x^2-2x=8$$

$$x^2-2x+1=9$$

$$x-1=\pm 3$$

$$x=4 \text{ or } -2$$

Substituting in (4), $y=\dfrac{12+4}{2}$ or $\dfrac{-6+4}{2}$

$$=8 \text{ or } -1$$

Art. 275.—Page 237.

4. $\begin{cases} x+y=1 & (1) \\ xy=-6 & (2) \end{cases}$

Squaring (1), $x^2+2xy+y^2=1$ (3)

From (2), $4xy=-24$ (4)

Subtracting (4) from (3), $x^2-2xy+y^2=25$

$x-y=\pm 5$ (5)

Adding (1) and (5), $2x=1\pm 5=6 \text{ or } -4$

$x=3 \text{ or } -2$

Subtracting (5) from (1), $2y=1\mp 5=-4 \text{ or } 6$

$y=-2 \text{ or } 3$

5. $\begin{cases} x-y=6 & (1) \\ x^2+y^2=90 & (2) \end{cases}$

Squaring (1), $x^2-2xy+y^2=36$ (3)

Subtracting (3) from (2), $2xy=54$ (4)

Adding (2) and (4), $x^2+2xy+y^2=144$

$x+y=\pm 12$ (5)

Adding (1) and (5), $2x=6\pm 12=18 \text{ or } -6$

$x=9 \text{ or } -3$

Subtracting (1) from (5), $2y=-6\pm 12=6 \text{ or } -18$

$y=3 \text{ or } -9$

6. $\begin{cases} x - y = -10 & (1) \\ xy = -21 & (2) \end{cases}$

Squaring (1), $x^2 - 2xy + y^2 = 100$ (3)

From (2), $4xy = -84$ (4)

Adding (3) and (4), $x^2 + 2xy + y^2 = 16$

$x + y = \pm 4$ (5)

Adding (1) and (5), $2x = -10 \pm 4 = -6 \text{ or } -14$

$x = -3 \text{ or } -7$

Subtracting (1) from (5), $2y = 10 \pm 4 = 14 \text{ or } 6$

$y = 7 \text{ or } 3$

7. $\begin{cases} x^3 + y^3 = -19 & (1) \\ x^2 - xy + y^2 = 19 & (2) \end{cases}$

Dividing (1) by (2), $x + y = -1$ (3)

Squaring (3), $x^2 + 2xy + y^2 = 1$ (4)

Subtracting (4) from (2), $-3xy = 18$

$-xy = 6$ (5)

Adding (2) and (5), $x^2 - 2xy + y^2 = 25$

$x - y = \pm 5$ (6)

Adding (3) and (6), $2x = -1 \pm 5 = 4 \text{ or } -6$

$x = 2 \text{ or } -3$

Subtracting (6) from (3), $2y = -1 \mp 5 = -6 \text{ or } 4$

$y = -3 \text{ or } 2$

8. $\begin{cases} x^2 + y^2 = 25 & (1) \\ xy = 12 & (2) \end{cases}$

From (2), $2xy = 24$ (3)

Adding (1) and (3), $x^2 + 2xy + y^2 = 49$

$x + y = \pm 7$ (4)

Subtracting (3) from (1), $x^2 - 2xy + y^2 = 1$

$x - y = \pm 1$ (5)

Adding (4) and (5), $2x = \pm 7 \pm 1$

$= 8, 6, -6, \text{ or } -8$

$x = \pm 4 \text{ or } \pm 3$

Subtracting (5) from (4), $2y = \pm 7 \mp 1$

$= 6, 8, -8, \text{ or } -6$

$y = \pm 3 \text{ or } \pm 4$

9. $\begin{cases} x + y = -4 & (1) \\ x^2 + y^2 = 58 & (2) \end{cases}$

Squaring (1), $x^2 + 2xy + y^2 = 16$ (3)

Subtracting (3) from (2), $-2xy = 42$ (4)

Adding (2) and (4), $x^2 - 2xy + y^2 = 100$

$$x - y = \pm 10 \qquad (5)$$

Adding (1) and (5), $2x = -4 \pm 10 = 6 \text{ or } -14$

$$x = 3 \text{ or } -7$$

Subtracting (5) from (1), $2y = -4 \mp 10 = -14 \text{ or } 6$

$$y = -7 \text{ or } 3$$

10. $$\begin{cases} x^3 - y^3 = 98 & (1) \\ x - y = 2 & (2) \end{cases}$$

Dividing (1) by (2), $x^2 + xy + y^2 = 49$ (3)

Squaring (2), $x^2 - 2xy + y^2 = 4$ (4)

Subtracting (4) from (3), $3xy = 45$

$$xy = 15 \qquad (5)$$

Adding (3) and (5), $x^2 + 2xy + y^2 = 64$

$$x + y = \pm 8 \qquad (6)$$

Adding (2) and (6), $2x = 2 \pm 8 = 10 \text{ or } -6$

$$x = 5 \text{ or } -3$$

Subtracting (2) from (6), $2y = -2 \pm 8 = 6 \text{ or } -10$

$$y = 3 \text{ or } -5$$

11. $$\begin{cases} x^3 + y^3 = 9 & (1) \\ x + y = 3 & (2) \end{cases}$$

Dividing (1) by (2), $x^2 - xy + y^2 = 3$ (3)

Squaring (2), $x^2 + 2xy + y^2 = 9$ (4)

Subtracting (4) from (3), $-3xy = -6$

$$-xy = -2 \qquad (5)$$

Adding (3) and (5), $x^2 - 2xy + y^2 = 1$

$$x - y = \pm 1 \qquad (6)$$

Adding (2) and (6), $2x = 3 \pm 1 = 4 \text{ or } 2$

$$x = 2 \text{ or } 1$$

Subtracting (6) from (2), $2y = 3 \mp 1 = 2 \text{ or } 4$

$$y = 1 \text{ or } 2$$

12. $$\begin{cases} x - y = \dfrac{1}{2} & (1) \\ xy = 60 & (2) \end{cases}$$

Squaring (1), $x^2 - 2xy + y^2 = \dfrac{1}{4}$ (3)

From (2), $4xy = 240$ (4)

Adding (3) and (4), $x^2 + 2xy + y^2 = \dfrac{961}{4}$

$$x + y = \pm \frac{31}{2} \qquad (5)$$

Adding (1) and (5), $2x = \frac{1}{2} \pm \frac{31}{2} = 16 \text{ or } -15$

$$x = 8 \text{ or } -\frac{15}{2}$$

Subtracting (1) from (5), $2y = -\frac{1}{2} \pm \frac{31}{2} = 15 \text{ or } -16$

$$y = \frac{15}{2} \text{ or } -8$$

13. $\begin{cases} x^2 + y^2 = 85 & (1) \\ xy = 42 & (2) \end{cases}$

From (2), $2xy = 84$ (3)

Adding (1) and (3), $x^2 + 2xy + y^2 = 169$

$x + y = \pm 13$ (4)

Subtracting (3) from (1), $x^2 - 2xy + y^2 = 1$

$x - y = \pm 1$ (5)

Adding (4) and (5), $2x = \pm 13 \pm 1$

$= 14, 12, -12, \text{ or } -14$

$x = \pm 7 \text{ or } \pm 6$

Subtracting (5) from (4), $2y = \pm 13 \mp 1$

$= 12, 14, -14, \text{ or } -12$

$y = \pm 6 \text{ or } \pm 7$

14. $\begin{cases} x^3 - y^3 = -316 & (1) \\ x - y = -4 & (2) \end{cases}$

Dividing (1) by (2), $x^2 + xy + y^2 = 79$ (3)

Squaring (2), $x^2 - 2xy + y^2 = 16$ (4)

Subtracting (4) from (3), $3xy = 63$

$xy = 21$ (5)

Adding (3) and (5), $x^2 + 2xy + y^2 = 100$

$x + y = \pm 10$ (6)

Adding (2) and (6), $2x = -4 \pm 10 = 6 \text{ or } -14$

$x = 3 \text{ or } -7$

Subtracting (2) from (6), $2y = 4 \pm 10 = 14 \text{ or } -6$

$y = 7 \text{ or } -3$

15. $\begin{cases} x^2 + y^2 = 193 & (1) \\ x + y = -5 & (2) \end{cases}$

Squaring (2), $x^2 + 2xy + y^2 = 25$ (3)

Subtracting (3) from (1), $-2xy = 168$ (4)

Adding (1) and (4), $x^2 - 2xy + y^2 = 361$

$x - y = \pm 19$ (5)

Adding (2) and (5), $2x = -5 \pm 19 = 14$ or -24

$x = 7$ or -12

Subtracting (5) from (2) $2y = -5 \mp 19 = -24$ or 14

$y = -12$ or 7

16. $\begin{cases} x + y = 12 & (1) \\ xy = -45 & (2) \end{cases}$

Squaring (1), $x^2 + 2xy + y^2 = 144$ (3)

From (2), $4xy = -180$ (4)

Subtracting (4) from (3), $x^2 - 2xy + y^2 = 324$

$x - y = \pm 18$ (5)

Adding (1) and (5), $2x = 12 \pm 18 = 30$ or -6

$x = 15$ or -3

Subtracting (5) from (1), $2y = 12 \mp 18 = -6$ or 30

$y = -3$ or 15

17. $\begin{cases} x^3 - y^3 = -65 & (1) \\ x^2 + xy + y^2 = 13 & (2) \end{cases}$

Dividing (1) by (2), $x - y = -5$ (3)

Squaring (3), $x^2 - 2xy + y^2 = 25$ (4)

Subtracting (4) from (2), $3xy = -12$

$xy = -4$ (5)

Adding (2) and (5), $x^2 + 2xy + y^2 = 9$

$x + y = \pm 3$ (6)

Adding (3) and (6), $2x = -5 \pm 3 = -2$ or -8

$x = -1$ or -4

Subtracting (3) from (6), $2y = 5 \pm 3 = 8$ or 2

$y = 4$ or 1

18. $\begin{cases} x^2 + y^2 = 157 & (1) \\ x - y = -5 & (2) \end{cases}$

Squaring (2), $x^2 - 2xy + y^2 = 25$ (3)

Subtracting (3) from (1), $2xy = 132$ (4)

Adding (1) and (4), $x^2 + 2xy + y^2 = 289$

$x + y = \pm 17$ (5)

Adding (2) and (5), $2x = -5 \pm 17 = 12$ or -22

$x = 6$ or -11

Subtracting (2) from (5), $2y = 5 \pm 17 = 22$ or -12

$y = 11$ or -6

19. $\begin{cases} x^3 + y^3 = -386 & (1) \\ x + y = -2 & (2) \end{cases}$

Dividing (1) by (2), $x^2 - xy + y^2 = 193$ (3)

Squaring (2), $x^2 + 2xy + y^2 = 4$ (4)

Subtracting (4) from (3), $-3xy = 189$

$$-xy = 63 \qquad (5)$$

Adding (3) and (5), $x^2 - 2xy + y^2 = 256$

$$x - y = \pm 16 \qquad (6)$$

Adding (2) and (6), $2x = -2 \pm 16 = 14 \text{ or } -18$

$$x = 7 \text{ or } -9$$

Subtracting (6) from (2), $2y = -2 \mp 16 = -18 \text{ or } 14$

$$y = -9 \text{ or } 7$$

Art. 276.—Page 239.

2.

$$\begin{cases} x^2 - xy = 35 & (1) \\ xy + y^2 = 18 & (2) \end{cases}$$

Putting $y = vx$ in (1) and (2),

$$x^2 - vx^2 = 35; \text{ or, } x^2 = \frac{35}{1-v} \qquad (3)$$

$$vx^2 + v^2x^2 = 18; \text{ or, } x^2 = \frac{18}{v+v^2}$$

Equating the values of x^2, $\dfrac{35}{1-v} = \dfrac{18}{v+v^2}$

$$35v + 35v^2 = 18 - 18v$$

$$35v^2 + 53v = 18$$

$$4 \times 35^2 v^2 + 140 \times 53v = 2520$$

$$70^2 v^2 + 140 \times 53v + 53^2 = 5329$$

$$70v + 53 = \pm 73$$

$$70v = 20 \text{ or } -126$$

$$v = \frac{2}{7} \text{ or } -\frac{9}{5}$$

Substituting in (3), $x^2 = \dfrac{35}{1-\frac{2}{7}} \text{ or } \dfrac{35}{1+\frac{9}{5}}$

$$= 49 \text{ or } \frac{25}{2}$$

Whence, $x = \pm 7 \text{ or } \pm \dfrac{5}{\sqrt{2}}$

$$= \pm 7 \text{ or } \pm \frac{5}{2}\sqrt{2}$$

Substituting these values in $y = vx$, $y = \dfrac{2}{7}(\pm 7) \text{ or } -\dfrac{9}{5} \ \pm \dfrac{5}{2}\sqrt{2})$

$$= \pm 2 \text{ or } \mp \frac{9}{2}\sqrt{2}$$

3. $$\begin{cases} 2x^2 + xy = 15 & (1) \\ x^2 - y^2 = 8 & (2) \end{cases}$$

Putting $y = vx$ in (1) and (2),

$$2x^2 + vx^2 = 15;\ \text{or},\ x^2 = \frac{15}{2+v} \qquad (3)$$

$$x^2 - v^2x^2 = 8;\ \text{or},\ x^2 = \frac{8}{1-v^2}$$

Equating the values of x^2, $$\frac{15}{2+v} = \frac{8}{1-v^2}$$

$$15 - 15v^2 = 16 + 8v$$
$$-15v^2 - 8v = 1$$
$$225v^2 + 120v = -15$$
$$225v^2 + 120v + 16 = 1$$
$$15v + 4 = \pm 1$$
$$15v = -5 \text{ or } -3$$
$$v = -\frac{1}{3} \text{ or } -\frac{1}{5}$$

Substituting in (3), $$x^2 = \frac{15}{2 - \frac{1}{3}} \text{ or } \frac{15}{2 - \frac{1}{5}}$$

$$= 9 \text{ or } \frac{25}{3}$$
$$x = \pm 3 \text{ or } \pm \frac{5}{\sqrt{3}}$$
$$= \pm 3 \text{ or } \pm \frac{5}{3}\sqrt{3}$$

Substituting these values in $y = vx$, $$y = -\frac{1}{3}(\pm 3) \text{ or } -\frac{1}{5}\left(\pm \frac{5}{3}\sqrt{3}\right)$$

$$= \mp 1 \text{ or } \mp \frac{1}{3}\sqrt{3}$$

4. $$\begin{cases} x^2 + xy - y^2 = -11 & (1) \\ x^2 + y^2 = 13 & (2) \end{cases}$$

Putting $y = vx$ in (1) and (2),

$$x^2 + vx^2 - v^2x^2 = -11;\ \text{or},\ x^2 = \frac{-11}{1+v-v^2}$$

$$x^2 + v^2x^2 = 13;\ \text{or},\ x^2 = \frac{13}{1+v^2} \qquad (3)$$

Equating the values of x^2, $$\frac{-11}{1+v-v^2} = \frac{13}{1+v^2}$$

$$-11 - 11v^2 = 13 + 13v - 13v^2$$
$$2v^2 - 13v = 24$$
$$16v^2 - 104v = 192$$

$$16v^2 - 104v + 169 = 361$$
$$4v - 13 = \pm 19$$
$$4v = -6 \text{ or } 32$$
$$v = -\frac{3}{2} \text{ or } 8$$

Substituting in (3), $$x^2 = \frac{13}{1+\frac{9}{4}} \text{ or } \frac{13}{1+64}$$
$$= 4 \text{ or } \frac{1}{5}$$
$$x = \pm 2 \text{ or } \pm \frac{1}{\sqrt{5}}$$
$$= \pm 2 \text{ or } \pm \frac{1}{5}\sqrt{5}$$

Substituting these values in $y = vx$, $$y = -\frac{3}{2}(\pm 2) \text{ or } 8(\pm \frac{1}{5}\sqrt{5})$$
$$= \mp 3 \text{ or } \pm \frac{8}{5}\sqrt{5}$$

5. $$\begin{cases} x^2 + xy + 4y^2 = 6 & (1) \\ 3x^2 + 8y^2 = 14 & (2) \end{cases}$$

Putting $y = vx$ in (1) and (2),

$$x^2 + vx^2 + 4v^2x^2 = 6; \text{ or, } x^2 = \frac{6}{1+v+4v^2}$$
$$3x^2 + 8v^2x^2 = 14; \text{ or, } x^2 = \frac{14}{3+8v^2} \qquad (3)$$

Equating the values of x^2, $$\frac{6}{1+v+4v^2} = \frac{14}{3+8v^2}$$
$$18 + 48v^2 = 14 + 14v + 56v^2$$
$$-8v^2 - 14v = -4$$
$$64v^2 + 112v = 32$$
$$64v^2 + 112v + 49 = 81$$
$$8v + 7 = \pm 9$$
$$8v = 2 \text{ or } -16$$
$$v = \frac{1}{4} \text{ or } -2$$

Substituting in (3), $$x^2 = \frac{14}{3+\frac{1}{2}} \text{ or } \frac{14}{3+32}$$
$$= 4 \text{ or } \frac{2}{5}$$
$$x = \pm 2 \text{ or } \pm \sqrt{\frac{2}{5}}$$
$$= \pm 2 \text{ or } \pm \frac{1}{5}\sqrt{10}$$

Substituting these values in $y = vx$, $\quad y = \frac{1}{4}(\pm 2)$ or $-2(\pm \frac{1}{5}\sqrt{10})$

$$= \pm\frac{1}{2} \text{ or } \mp\frac{2}{5}\sqrt{10}$$

6. $$\begin{cases} x^2 + xy = 12 & (1) \\ xy - y^2 = 2 & (2) \end{cases}$$

Putting $y = vx$ in (1) and (2),

$$x^2 + vx^2 = 12; \text{ or, } x^2 = \frac{12}{1+v} \qquad (3)$$

$$vx^2 - v^2x^2 = 2; \text{ or, } x^2 = \frac{2}{v - v^2}$$

Equating the values of x^2, $\quad \frac{12}{1+v} = \frac{2}{v-v^2}$

$$12v - 12v^2 = 2 + 2v$$

$$-12v^2 + 10v = 2$$

$$144v^2 - 120v = -24$$

$$144v^2 - 120v + 25 = 1$$

$$12v - 5 = \pm 1$$

$$12v = 4 \text{ or } 6$$

$$v = \frac{1}{3} \text{ or } \frac{1}{2}$$

Substituting in (3), $\quad x^2 = \frac{12}{1+\frac{1}{3}}$ or $\frac{12}{1+\frac{1}{2}}$

$$= 9 \text{ or } 8$$

$$x = \pm 3 \text{ or } \pm 2\sqrt{2}$$

Substituting these values in $y = vx$,

$$y = \frac{1}{3}(\pm 3) \text{ or } \frac{1}{2}(\pm 2\sqrt{2})$$

$$= \pm 1 \text{ or } \pm\sqrt{2}$$

7. $$\begin{cases} 2y^2 - 4xy + 3x^2 = 17 & (1) \\ y^2 - x^2 = 16 & (2) \end{cases}$$

Putting $y = vx$ in (1) and (2),

$$2v^2x^2 - 4vx^2 + 3x^2 = 17; \text{ or, } x^2 = \frac{17}{2v^2 - 4v + 3}$$

$$v^2x^2 - x^2 = 16; \text{ or, } x^2 = \frac{16}{v^2 - 1} \qquad (3)$$

Equating the values of x^2, $\quad \frac{17}{2v^2 - 4v + 3} = \frac{16}{v^2 - 1}$

$$17v^2 - 17 = 32v^2 - 64v + 48$$

$$-15v^2 + 64v = 65$$

$$225v^2 - 960v = -975$$

$$225v^2 - 960v + 1024 = 49$$
$$15v - 32 = \pm 7$$
$$15v = 25 \text{ or } 39$$
$$v = \frac{5}{3} \text{ or } \frac{13}{5}$$

Substituting in (3), $$x^2 = \frac{16}{\frac{25}{9} - 1} \text{ or } \frac{16}{\frac{169}{25} - 1}$$
$$= 9 \text{ or } \frac{25}{9}$$
$$x = \pm 3 \text{ or } \pm \frac{5}{3}$$

Substituting these values in $y = vx$
$$y = \frac{5}{3}(\pm 3) \text{ or } \frac{13}{5}\left(\pm \frac{5}{3}\right)$$
$$= \pm 5 \text{ or } \pm \frac{13}{3}$$

8. $$\begin{cases} 2x^2 - 2xy - y^2 = 3 & (1) \\ x^2 + 3xy + y^2 = 11 & (2) \end{cases}$$

Putting $y = vx$ in (1) and (2),
$$2x^2 - 2vx^2 - v^2x^2 = 3; \text{ or, } x^2 = \frac{3}{2 - 2v - v^2}$$
$$x^2 + 3vx^2 + v^2x^2 = 11; \text{ or, } x^2 = \frac{11}{1 + 3v + v^2} \quad (3)$$

Equating the values of x^2, $$\frac{3}{2 - 2v - v^2} = \frac{11}{1 + 3v + v^2}$$
$$3 + 9v + 3v^2 = 22 - 22v - 11v^2$$
$$14v^2 + 31v = 19$$
$$4 \times 14^2v^2 + 56 \times 31v = 1064$$
$$28^2v^2 + 56 \times 31v + 961 = 2025$$
$$28v + 31 = \pm 45$$
$$28v = 14 \text{ or } -76$$
$$v = \frac{1}{2} \text{ or } -\frac{19}{7}$$

Substituting in (3), $$x^2 = \frac{11}{1 + \frac{3}{2} + \frac{1}{4}} \text{ or } \frac{11}{1 - \frac{57}{7} + \frac{361}{49}}$$
$$= 4 \text{ or } 49$$
$$x = \pm 2 \text{ or } \pm 7$$

Substituting these values in $y = vx$, $$y = \frac{1}{2}(\pm 2) \text{ or } -\frac{19}{7}(\pm 7)$$
$$= \pm 1 \text{ or } \mp 19$$

9. $$\begin{cases} 6x^2-5xy+2y^2=12 & (1) \\ 3x^2+2xy-3y^2=-3 & (2) \end{cases}$$

Putting $y=vx$ in (1) and (2),

$$6x^2-5vx^2+2v^2x^2=12;\ \text{or, } x^2=\frac{12}{6-5v+2v^2}$$

$$3x^2+2vx^2-3v^2x^2=-3;\ \text{or, } x^2=\frac{-3}{3+2v-3v^2} \qquad (3)$$

Equating the values of x^2, $\dfrac{12}{6-5v+2v^2}=\dfrac{-3}{3+2v-3v^2}$

$$36+24v-36v^2=-18+15v-6v^2$$

$$-30v^2+9v=-54$$

$$10v^2-3v=18$$

$$400v^2-120v=720$$

$$400v^2-120v+9=729$$

$$20v-3=\pm 27$$

$$20v=30 \text{ or } -24$$

$$v=\frac{3}{2} \text{ or } -\frac{6}{5}$$

Substituting in (3), $x^2=\dfrac{-3}{3+3-\frac{27}{4}} \text{ or } \dfrac{-3}{3-\frac{12}{5}-\frac{108}{25}}$

$$=4 \text{ or } \frac{25}{31}$$

$$x=\pm 2 \text{ or } \pm\frac{5}{\sqrt{31}}$$

$$=\pm 2 \text{ or } \pm\frac{5}{31}\sqrt{31}$$

Substituting these values in $y=vx$, $y=\frac{3}{2}(\pm 2) \text{ or } -\frac{6}{5}(\pm\frac{5}{31}\sqrt{31})$

$$=\pm 3 \text{ or } \mp\frac{6}{31}\sqrt{31}$$

10. $$\begin{cases} x^2+xy-y^2=1 & (1) \\ x^2-xy+2y^2=8 & (2) \end{cases}$$

Putting $y=vx$ in (1) and (2),

$$x^2+vx^2-v^2x^2=1;\ \text{or, } x^2=\frac{1}{1+v-v^2} \qquad (3)$$

$$x^2-vx^2+2v^2x^2=8;\ \text{or, } x^2=\frac{8}{1-v+2v^2}$$

Equating the values of x^2, $\dfrac{1}{1+v-v^2}=\dfrac{8}{1-v+2v^2}$

$$1 - v + 2v^2 = 8 + 8v - 8v^2$$
$$10v^2 - 9v = 7$$
$$400v^2 - 360v = 280$$
$$400v^2 - 360v + 81 = 361$$
$$20v - 9 = \pm 19$$
$$20v = -10 \text{ or } 28$$
$$v = -\frac{1}{2} \text{ or } \frac{7}{5}$$

Substituting in (3), $$x^2 = \frac{1}{1 - \frac{1}{2} - \frac{1}{4}} \text{ or } \frac{1}{1 + \frac{7}{5} - \frac{49}{25}}$$
$$= 4 \text{ or } \frac{25}{11}$$
$$x = \pm 2 \text{ or } \pm \frac{5}{\sqrt{11}}$$
$$= \pm 2 \text{ or } \pm \frac{5}{11}\sqrt{11}$$

Substituting these values in $y = vx$, $$y = -\frac{1}{2}(\pm 2) \text{ or } \frac{7}{5}(\pm \frac{5}{11}\sqrt{11})$$
$$= \mp 1 \text{ or } \pm \frac{7}{11}\sqrt{11}$$

11. $$\begin{cases} 4xy - x^2 = 5 & (1) \\ 13x^2 - 31xy + 16y^2 = 2\frac{1}{2} & (2) \end{cases}$$

From (2), $$26x^2 - 62xy + 32y^2 = 5 \qquad (3)$$

Putting $y = vx$ in (1) and (3),
$$4vx^2 - x^2 = 5;\ \text{or, } x^2 = \frac{5}{4v - 1} \qquad (4)$$
$$26x^2 - 62vx^2 + 32v^2x^2 = 5;\ \text{or, } x^2 = \frac{5}{26 - 62v + 32v^2}$$

Equating the values of x^2, $$\frac{5}{4v - 1} = \frac{5}{26 - 62v + 32v^2}$$
$$\frac{1}{4v - 1} = \frac{1}{26 - 62v + 32v^2}$$
$$26 - 62v + 32v^2 = 4v - 1$$
$$32v^2 - 66v = -27$$
$$1024v^2 - 32 \times 66v = -864$$
$$1024v^2 - 32 \times 66v + 1089 = 225$$
$$32v - 33 = \pm 15$$
$$32v = 48 \text{ or } 18$$
$$v = \frac{3}{2} \text{ or } \frac{9}{16}$$

Substituting in (4), $\quad x^2 = \dfrac{5}{6-1}$ or $\dfrac{5}{\frac{9}{4}-1}$

$= 1$ or 4

$x = \pm 1$ or ± 2

Substituting these values in $y = vx$, $\quad y = \frac{3}{2}(\pm 1)$ or $\frac{9}{16}(\pm 2)$

$= \pm\frac{3}{2}$ or $\pm\frac{9}{8}$

Art. 277.—Pages 242, 243.

5.
$$\begin{cases} xy - 2x = 5 & (1) \\ xy + 3y = -2 & (2) \end{cases}$$

Subtracting (1) from (2), $\quad 2x + 3y = -7$

$3y = -2x - 7$

$y = -\dfrac{2x+7}{3} \qquad (3)$

Substituting in (1), $\quad -x\left(\dfrac{2x+7}{3}\right) - 2x = 5$

$-2x^2 - 7x - 6x = 15$

$2x^2 + 13x = -15$

$16x^2 + 104x + 169 = 49$

$4x + 13 = \pm 7$

$4x = -20$ or -6

$x = -5$ or $-\frac{3}{2}$

Substituting in (3), $\quad y = -\dfrac{-10+7}{3}$ or $-\dfrac{-3+7}{3}$

$= 1$ or $-\frac{4}{3}$

6.
$$\begin{cases} x + y = 9 & (1) \\ x^{\frac{1}{3}} + y^{\frac{1}{3}} = 3 & (2) \end{cases}$$

Dividing (1) by (2), $\quad x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = 3 \qquad (3)$

Squaring (2), $\quad x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = 9 \qquad (4)$

Subtracting (4) from (3), $\quad -3x^{\frac{1}{3}}y^{\frac{1}{3}} = -6$

$-x^{\frac{1}{3}}y^{\frac{1}{3}} = -2 \qquad (5)$

Adding (3) and (5), $x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = 1$

$$x^{\frac{1}{3}} - y^{\frac{1}{3}} = \pm 1 \qquad (6)$$

Adding (2) and (6), $2x^{\frac{1}{3}} = 3 \pm 1 = 4 \text{ or } 2$

$$x^{\frac{1}{3}} = 2 \text{ or } 1$$

$$x = 8 \text{ or } 1$$

Subtracting (6) from (2) $2y^{\frac{1}{3}} = 3 \mp 1 = 2 \text{ or } 4$

$$y^{\frac{1}{3}} = 1 \text{ or } 2$$

$$y = 1 \text{ or } 8$$

7. $$\begin{cases} 4x^2 - 3y^2 = -11 & (1) \\ 11x^2 + 5y^2 = 301 & (2) \end{cases}$$

Multiplying (1) by 5 and (2) by 3,

$$\begin{array}{r} 20x^2 - 15y^2 = -55 \\ \underline{33x^2 + 15y^2 = 903} \end{array}$$

Adding, $53x^2 = 848$

$$x^2 = 16$$

$$x = \pm 4$$

Substituting $x = \pm 4$ in (1), $64 - 3y^2 = -11$

$$-3y^2 = -75$$

$$y^2 = 25$$

$$y = \pm 5$$

8. $$\begin{cases} x^3 + y^3 = 35 & (1) \\ x^2y + xy^2 = 30 & (2) \end{cases}$$

Multiplying (2) by 3, $3x^2y + 3xy^2 = 90$ (3)

Adding (1) and (3),

$$x^3 + 3x^2y + 3xy^2 + y^3 = 125$$

Extracting the cube root, $x + y = 5$ (4)

Dividing (2) by (4), $xy = 6$ (5)

Squaring (4), $x^2 + 2xy + y^2 = 25$ (6)

From (5), $4xy = 24$ (7)

Subtracting (7) from (6), $x^2 - 2xy + y^2 = 1$

$$x - y = \pm 1 \qquad (8)$$

Adding (4) and (8), $2x = 5 \pm 1 = 6 \text{ or } 4$

$$x = 3 \text{ or } 2$$

Subtracting (8) from (4), $2y = 5 \mp 1 = 4 \text{ or } 6$

$$y = 2 \text{ or } 3$$

9.
$$\begin{cases} \dfrac{x}{y}+\dfrac{y}{x}=\dfrac{29}{10} & (1) \\ 3x-2y=4 & (2) \end{cases}$$

From (1), $$10x^2+10y^2=29xy \qquad (3)$$

From (2), $$3x-4=2y$$

or, $$y=\frac{3x-4}{2} \qquad (4)$$

Substituting from (4) in (3),
$$10x^2+10\left(\frac{9x^2-24x+16}{4}\right)=29x\left(\frac{3x-4}{2}\right)$$
$$20x^2+45x^2-120x+80=87x^2-116x$$
$$-22x^2-4x=-80$$
$$11x^2+2x=40$$
$$121x^2+22x+1=441$$
$$11x+1=\pm 21$$
$$11x=-22 \text{ or } 20$$
$$x=-2 \text{ or } \frac{20}{11}$$

Substituting in (4), $$y=\frac{-6-4}{2} \text{ or } \frac{\frac{60}{11}-4}{2}$$
$$=-5 \text{ or } \frac{8}{11}$$

10.
$$\begin{cases} x^2+y^2=5m^2 & (1) \\ x-y=m & (2) \end{cases}$$

Squaring (2), $$x^2-2xy+y^2=m^2 \qquad (3)$$

Subtracting (3) from (1), $$2xy=4m^2 \qquad (4)$$

Adding (1) and (4), $$x^2+2xy+y^2=9m^2$$
$$x+y=\pm 3m \qquad (5)$$

Adding (2) and (5), $$2x=m\pm 3m=4m \text{ or } -2m$$
$$x=2m \text{ or } -m$$

Subtracting (2) from (5), $$2y=-m\pm 3m=2m \text{ or } -4m$$
$$y=m \text{ or } -2m$$

11.
$$\begin{cases} x^2+y^2+x+y=18 & (1) \\ xy=6 & (2) \end{cases}$$

From (2), $$2xy=12 \qquad (3)$$

Adding (1) and (3), $$x^2+2xy+y^2+x+y=30$$
$$(x+y)^2+(x+y)=30$$
$$4(x+y)^2+4(x+y)+1=121$$
$$2(x+y)+1=\pm 11$$
$$2(x+y)=10 \text{ or } -12$$
$$x+y=5 \text{ or } -6 \qquad (4)$$

Squaring (4), $x^2 + 2xy + y^2 = 25 \text{ or } 36$ (5)

From (2), $4xy = 24$ (6)

Subtracting (6) from (5), $x^2 - 2xy + y^2 = 1 \text{ or } 12$

$$x - y = \pm 1 \text{ or } \pm 2\sqrt{3} \quad (7)$$

Adding (4) and (7), $2x = 5 \pm 1 \text{ or } -6 \pm 2\sqrt{3}$

$$x = 3, 2, \text{ or } -3 \pm \sqrt{3}$$

Subtracting (7) from (4), $2y = 5 \mp 1 \text{ or } -6 \mp 2\sqrt{3}$

$$y = 2, 3, \text{ or } -3 \mp \sqrt{3}$$

12. $\begin{cases} x^2 - 2xy = 16 & (1) \\ 2xy + y^2 = -3 & (2) \end{cases}$

Putting $y = vx$ in (1) and (2),

$$x^2 - 2vx^2 = 16; \text{ or, } x^2 = \frac{16}{1-2v} \quad (3)$$

$$2vx^2 + v^2x^2 = -3; \text{ or, } x^2 = \frac{-3}{2v+v^2}$$

Equating the values of x^2, $\dfrac{16}{1-2v} = \dfrac{-3}{2v+v^2}$

$$32v + 16v^2 = -3 + 6v$$

$$16v^2 + 26v = -3$$

$$256v^2 + 416v + 169 = 121$$

$$16v + 13 = \pm 11$$

$$16v = -24 \text{ or } -2$$

$$v = -\frac{3}{2} \text{ or } -\frac{1}{8}$$

Substituting in (3), $x^2 = \dfrac{16}{1+3} \text{ or } \dfrac{16}{1+\frac{1}{4}}$

$$= 4 \text{ or } \frac{64}{5}$$

$$x = \pm 2 \text{ or } \pm \frac{8}{\sqrt{5}}$$

$$= \pm 2 \text{ or } \pm \frac{8}{5}\sqrt{5}$$

Substituting these values in $y = vx$, $y = -\dfrac{3}{2}(\pm 2) \text{ or } -\dfrac{1}{8}(\pm \dfrac{8}{5}\sqrt{5})$

$$= \mp 3 \text{ or } \mp \frac{1}{5}\sqrt{5}$$

13. $\begin{cases} x^3 + y^3 = 18xy \\ x + y = 12 \end{cases}$

Putting $x = u + v$ and $y = u - v$,

$$(u+v)^3 + (u-v)^3 = 18(u^2 - v^2) \quad (1)$$

$$u + v + u - v = 12 \quad (2)$$

Reducing (2), $2u = 12$

$$u = 6$$

Reducing (1), $2u^3 + 6uv^2 = 18(u^2 - v^2)$

Substituting the value of u, $432 + 36v^2 = 18(36 - v^2)$

$$24 + 2v^2 = 36 - v^2$$
$$3v^2 = 12$$
$$v^2 = 4$$
$$v = \pm 2$$

Therefore, $x = u + v = 6 \pm 2 = 8 \text{ or } 4$

and $y = u - v = 6 \mp 2 = 4 \text{ or } 8$

14. $$\begin{cases} x^2 + 3xy = -14 & (1) \\ xy + 4y^2 = 30 & (2) \end{cases}$$

Putting $y = vx$ in (1) and (2),

$$x^2 + 3vx^2 = -14; \text{ or, } x^2 = \frac{-14}{1+3v} \qquad (3)$$

$$vx^2 + 4v^2x^2 = 30; \text{ or, } x^2 = \frac{30}{v + 4v^2}$$

Equating the values of x^2, $\dfrac{-14}{1+3v} = \dfrac{30}{v+4v^2}$

$$-14v - 56v^2 = 30 + 90v$$
$$-56v^2 - 104v = 30$$
$$28v^2 + 52v = -15$$
$$28^2v^2 + 28 \times 52v + 676 = 256$$
$$28v + 26 = \pm 16$$
$$28v = -42 \text{ or } -10$$
$$v = -\frac{3}{2} \text{ or } -\frac{5}{14}$$

Substituting in (3), $x^2 = \dfrac{-14}{1 - \frac{9}{2}} \text{ or } \dfrac{-14}{1 - \frac{15}{14}}$

$$= 4 \text{ or } 196$$
$$x = \pm 2 \text{ or } \pm 14$$

Substituting these values in $y = vx$, $y = -\frac{3}{2}(\pm 2) \text{ or } -\frac{5}{14}(\pm 14)$

$$= \mp 3 \text{ or } \mp 5$$

15. $$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 11 & (1) \\ \dfrac{1}{xy} = 18 & (2) \end{cases}$$

Squaring (1), $\dfrac{1}{x^2} + \dfrac{2}{xy} + \dfrac{1}{y^2} = 121 \qquad (3)$

From (2), $$\frac{4}{xy} = 72 \qquad (4)$$

Subtracting (4) from (3), $$\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = 49$$

$$\frac{1}{x} - \frac{1}{y} = \pm 7 \qquad (5)$$

Adding (1) and (5), $$\frac{2}{x} = 11 \pm 7 = 18 \text{ or } 4$$

$$\frac{1}{x} = 9 \text{ or } 2$$

$$x = \frac{1}{9} \text{ or } \frac{1}{2}$$

Subtracting (5) from (1), $$\frac{2}{y} = 11 \mp 7 = 4 \text{ or } 18$$

$$\frac{1}{y} = 2 \text{ or } 9$$

$$y = \frac{1}{2} \text{ or } \frac{1}{9}$$

16. $$\begin{cases} x - y = a - b & (1) \\ xy = 2a^2 + 2ab & (2) \end{cases}$$

Squaring (1), $$x^2 - 2xy + y^2 = a^2 - 2ab + b^2 \qquad (3)$$

From (2), $$4xy = 8a^2 + 8ab \qquad (4)$$

Adding (3) and (4), $$x^2 + 2xy + y^2 = 9a^2 + 6ab + b^2$$

$$x + y = \pm(3a + b) \qquad (5)$$

Adding (1) and (5), $$2x = a - b \pm (3a + b)$$

$$= 4a \text{ or } -2a - 2b$$

$$x = 2a \text{ or } -a - b$$

Subtracting (1) from (5), $$2y = -a + b \pm (3a + b)$$

$$= 2a + 2b \text{ or } -4a$$

$$y = a + b \text{ or } -2a$$

17. $$\begin{cases} x^2 + y^2 = 9 - x & (1) \\ x^2 - y^2 = 6 & (2) \end{cases}$$

Adding (1) and (2), $$2x^2 = 15 - x$$

$$2x^2 + x = 15$$

$$16x^2 + 8x + 1 = 121$$

$$4x + 1 = \pm 11$$

$$4x = 10 \text{ or } -12$$

$$x = \frac{5}{2} \text{ or } -3$$

Substituting $x=\frac{5}{2}$ in (2), $\frac{25}{4}-y^2=6$

$$y^2=\frac{1}{4}$$

$$y=\pm\frac{1}{2}$$

Substituting $x=-3$ in (2), $9-y^2=6$

$$y^2=3$$

$$y=\pm\sqrt{3}$$

18. $\begin{cases}\dfrac{1}{x^2}+\dfrac{1}{y^2}=65 & (1)\\ \dfrac{1}{x}-\dfrac{1}{y}=11 & (2)\end{cases}$

Squaring (2), $\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=121$ (3)

Subtracting (3) from (1), $\frac{2}{xy}=-56$ (4)

Adding (1) and (4), $\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=9$

$$\frac{1}{x}+\frac{1}{y}=\pm 3 \quad (5)$$

Adding (2) and (5), $\frac{2}{x}=11\pm 3=14 \text{ or } 8$

$$\frac{1}{x}=7 \text{ or } 4$$

$$x=\frac{1}{7} \text{ or } \frac{1}{4}$$

Subtracting (2) from (5), $\frac{2}{y}=-11\pm 3=-8 \text{ or } -14$

$$\frac{1}{y}=-4 \text{ or } -7$$

$$y=-\frac{1}{4} \text{ or } -\frac{1}{7}$$

19. $\begin{cases}x^2+y^2-x-y=18 & (1)\\ xy+x+y=19 & (2)\end{cases}$

From (2), $2xy+2x+2y=38$ (3)

Adding (1) and (3), $x^2+2xy+y^2+x+y=56$

$$(x+y)^2+(x+y)=56$$

$$4(x+y)^2+4(x+y)+1=225$$

$$2(x+y)+1=\pm 15$$

$$2(x+y)=14 \text{ or } -16$$

$$x+y=7 \text{ or } -8 \quad (4)$$

From (2), $xy = 19 - (x + y)$

$= 19 - 7$ or $19 + 8$

$= 12$ or 27 (5)

Squaring (4), $x^2 + 2xy + y^2 = 49$ or 64 (6)

From (5), $4xy = 48$ or 108 (7)

Subtracting (7) from (6), $x^2 - 2xy + y^2 = 1$ or -44

$x - y = \pm 1$ or $\pm 2\sqrt{-11}$ (8)

Adding (4) and (8), $2x = 7 \pm 1$ or $-8 \pm 2\sqrt{-11}$

$x = 4, 3$, or $-4 \pm \sqrt{-11}$

Subtracting (8) from (4). $2y = 7 \mp 1$ or $-8 \mp 2\sqrt{-11}$

$y = 3, 4$, or $-4 \mp \sqrt{-11}$

20. $\begin{cases} x^2 + xy + y^2 = 7 & (1) \\ x^4 + x^2y^2 + y^4 = 133 & (2) \end{cases}$

Dividing (2) by (1), $x^2 - xy + y^2 = 19$ (3)

Adding (1) and (3), $2x^2 + 2y^2 = 26$

$x^2 + y^2 = 13$ (4)

Subtracting (3) from (1), $2xy = -12$ (5)

Adding (4) and (5), $x^2 + 2xy + y^2 = 1$

$x + y = \pm 1$ (6)

Subtracting (5) from (4), $x^2 - 2xy + y^2 = 25$

$x - y = \pm 5$ (7)

Adding (6) and (7), $2x = \pm 1 \pm 5$

$= \pm 6$ or ± 4

$x = \pm 3$ or ± 2

Subtracting (7) from (6), $2y = \pm 1 \mp 5$

$= \mp 4$ or ∓ 6

$y = \mp 2$ or ∓ 3

21. $\begin{cases} x^2y + xy^2 = 6 & (1) \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{2}{3} & (2) \end{cases}$

From (1), $xy(x + y) = 6$ (3)

From (2), $3(x + y) = 2xy$ (4)

Dividing (3) by (4), $\dfrac{xy}{3} = \dfrac{3}{xy}$

$x^2y^2 = 9$

$xy = \pm 3$ (5)

Substituting in (4), $3(x + y) = \pm 6$

$x + y = \pm 2$ (6)

Squaring (6), $x^2 + 2xy + y^2 = 4$ (7)

From (5), $4xy = \pm 12$ (8)

Subtracting (8) from (7), $x^2 - 2xy + y^2 = -8$ or 16

$$x - y = \pm 2\sqrt{-2} \text{ or } \pm 4 \qquad (9)$$

Adding (6) and (9), $2x = 2 \pm 2\sqrt{-2}$ or -2 ± 4

$$x = 1 \pm \sqrt{-2},\ 1, \text{ or } -3$$

Subtracting (9) from (6), $2y = 2 \mp 2\sqrt{-2}$ or -2 ∓ 4

$$y = 1 \mp \sqrt{-2},\ -3, \text{ or } 1$$

22. $\begin{cases} x^4 + y^4 = 17 \\ x - y = 3 \end{cases}$

Putting $x = u + v$ and $y = u - v$,

$$(u+v)^4 + (u-v)^4 = 17 \qquad (1)$$

$$u + v - (u - v) = 3 \qquad (2)$$

Reducing (2), $2v = 3$

$$v = \frac{3}{2}$$

Reducing (1), $2u^4 + 12u^2v^2 + 2v^4 = 17$

Substituting the value of v,

$$2u^4 + 27u^2 + \frac{81}{8} = 17$$

$$16u^4 + 216u^2 = 55$$

$$16u^4 + 216u^2 + 729 = 784$$

$$4u^2 + 27 = \pm 28$$

$$4u^2 = 1 \text{ or } -55$$

$$u^2 = \frac{1}{4} \text{ or } -\frac{55}{4}$$

$$u = \pm\frac{1}{2} \text{ or } \pm\frac{1}{2}\sqrt{-55}$$

Therefore, $x = u + v$

$$= \frac{3}{2} \pm \frac{1}{2} \text{ or } \frac{3}{2} \pm \frac{1}{2}\sqrt{-55}$$

$$= 2,\ 1, \text{ or } \frac{3 \pm \sqrt{-55}}{2}$$

and $y = u - v$

$$= -\frac{3}{2} \pm \frac{1}{2} \text{ or } -\frac{3}{2} \pm \frac{1}{2}\sqrt{-55}$$

$$= -1,\ -2, \text{ or } \frac{-3 \pm \sqrt{-55}}{2}$$

23. $\begin{cases} x^3 - y^3 = 7a^3 & (1) \\ x - y = a & (2) \end{cases}$

Dividing (1) by (2), $x^2 + xy + y^2 = 7a^2 \qquad (3)$

Squaring (2), $x^2 - 2xy + y^2 = a^2 \qquad (4)$

Subtracting (4) from (3), $3xy = 6a^2$

$xy = 2a^2$ (5)

Adding (3) and (5), $x^2 + 2xy + y^2 = 9a^2$

$x + y = \pm 3a$ (6)

Adding (2) and (6), $2x = a \pm 3a = 4a$ or $-2a$

$x = 2a$ or $-a$

Subtracting (2) from (6), $2y = -a \pm 3a = 2a$ or $-4a$

$y = a$ or $-2a$

24.
$$\begin{cases} x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 19 & (1) \\ x^2 + xy + y^2 = 133 & (2) \end{cases}$$

Dividing (2) by (1), $x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 7$ (3)

Adding (1) and (3), $2x + 2y = 26$

$x + y = 13$ (4)

Subtracting (3) from (1), $2x^{\frac{1}{2}}y^{\frac{1}{2}} = 12$

$x^{\frac{1}{2}}y^{\frac{1}{2}} = 6$

$xy = 36$ (5)

Squaring (4), $x^2 + 2xy + y^2 = 169$ (6)

From (5), $4xy = 144$ (7)

Subtracting (7) from (6), $x^2 - 2xy + y^2 = 25$

$x - y = \pm 5$ (8)

Adding (4) and (8), $2x = 13 \pm 5 = 18$ or 8

$x = 9$ or 4

Subtracting (8) from (4), $2y = 13 \mp 5 = 8$ or 18

$y = 4$ or 9

25.
$$\begin{cases} x + 2y = 3a + b & (1) \\ xy + y^2 = 2a(a + b) & (2) \end{cases}$$

From (1), $x = 3a + b - 2y$ (3)

Substituting in (2), $y(3a + b - 2y) + y^2 = 2a(a + b)$

$$(3a + b)y - 2y^2 + y^2 = 2a(a + b)$$
$$y^2 - (3a + b)y = -2a(a + b)$$
$$4y^2 - 4(3a + b)y = -8a(a + b)$$
$$4y^2 - 4(3a + b)y + (3a + b)^2 = (3a + b)^2 - 8a(a + b)$$
$$= a^2 - 2ab + b^2$$
$$2y - (3a + b) = \pm(a - b)$$
$$2y = 3a + b \pm (a - b)$$
$$= 4a \text{ or } 2a + 2b$$
$$y = 2a \text{ or } a + b$$

Substituting in (3), $x = 3a + b - 4a$

or $3a + b - 2(a + b)$

$= b - a$ or $a - b$

26. $\begin{cases} x^2y + xy^2 = 30 & (1) \\ x^4y^2 + x^2y^4 = 468 & (2) \end{cases}$

Squaring (1), $x^4y^2 + 2x^3y^3 + x^2y^4 = 900$ (3)

Subtracting (2) from (3), $2x^3y^3 = 432$

$x^3y^3 = 216$

$xy = 6$ (4)

Substituting in (1), $6x + 6y = 30$

$x + y = 5$ (5)

Squaring (5), $x^2 + 2xy + y^2 = 25$ (6)

From (4), $4xy = 24$ (7)

Subtracting (7) from (6), $x^2 - 2xy + y^2 = 1$

$x - y = \pm 1$ (8)

Adding (5) and (8), $2x = 5 \pm 1 = 6$ or 4

$x = 3$ or 2

Subtracting (8) from (5), $2y = 5 \mp 1 = 4$ or 6

$y = 2$ or 3

27. $\begin{cases} x^3 - y^3 = 6a^2b + 2b^3 & (1) \\ xy(x - y) = 2a^2b - 2b^3 & (2) \end{cases}$

From (2), $3x^2y - 3xy^2 = 6a^2b - 6b^3$ (3)

Subtracting (3) from (1),

$x^3 - 3x^2y + 3xy^2 - y^3 = 8b^3$

$x - y = 2b$ (4)

Dividing (2) by (4), $xy = a^2 - b^2$ (5)

Squaring (4), $x^2 - 2xy + y^2 = 4b^2$ (6)

From (5), $4xy = 4a^2 - 4b^2$ (7)

Adding (6) and (7), $x^2 + 2xy + y^2 = 4a^2$

$x + y = \pm 2a$ (8)

Adding (4) and (8), $2x = 2b \pm 2a$

$x = b \pm a$

Subtracting (4) from (8), $2y = -2b \pm 2a$

$y = -b \pm a$

28. $\begin{cases} x^2 + 3x + y = 73 - 2xy & (1) \\ y^2 + 3y + x = 44 & (2) \end{cases}$

Adding (1) and (2), $x^2 + y^2 + 4x + 4y = 117 - 2xy$

$(x + y)^2 + 4(x + y) = 117$

$(x + y)^2 + 4(x + y) + 4 = 121$

$x + y + 2 = \pm 11$

$x + y = 9$ or -13

$x = 9 - y$ or $-13 - y$

Substituting $x = 9 - y$ in (2),

$$y^2 + 3y + 9 - y = 44$$
$$y^2 + 2y = 35$$
$$y^2 + 2y + 1 = 36$$
$$y + 1 = \pm 6$$
$$y = 5 \text{ or } -7$$

Therefore,

$$x = 9 - y$$
$$= 4 \text{ or } 16$$

Substituting $x = -13 - y$ in (2),

$$y^2 + 3y - 13 - y = 44$$
$$y^2 + 2y = 57$$
$$y^2 + 2y + 1 = 58$$
$$y + 1 = \pm\sqrt{58}$$
$$y = -1 \pm \sqrt{58}$$

Therefore

$$x = -13 - y$$
$$= -13 + 1 \mp \sqrt{58}$$
$$= -12 \mp \sqrt{58}$$

29.
$$\begin{cases} \dfrac{x}{y} + \dfrac{4\sqrt{x}}{\sqrt{y}} = \dfrac{33}{4} & (1) \\ x - y = 5 & (2) \end{cases}$$

From (1),

$$\frac{x}{y} + \frac{4\sqrt{x}}{\sqrt{y}} + 4 = \frac{49}{4}$$
$$\frac{\sqrt{x}}{\sqrt{y}} + 2 = \pm\frac{7}{2}$$
$$\frac{\sqrt{x}}{\sqrt{y}} = \frac{3}{2} \text{ or } -\frac{11}{2}$$
$$\frac{x}{y} = \frac{9}{4} \text{ or } \frac{121}{4} \qquad (3)$$

From (2),

$$y = x - 5$$

Substituting in (3),

$$\frac{x}{x-5} = \frac{9}{4} \text{ or } \frac{121}{4}$$

Taking the first value,

$$4x = 9x - 45$$
$$-5x = -45$$
$$x = 9$$

and

$$y = x - 5 = 4$$

Taking the second value,

$$4x = 121x - 605$$
$$-117x = -605$$
$$x = \frac{605}{117} = 5\tfrac{20}{117}$$

and

$$y = x - 5 = \frac{20}{117}$$

30. $$\begin{cases} x^2 - xy + y^2 = 19 & (1) \\ 2x^2 - y^2 = -17 & (2) \end{cases}$$

Putting $y = vx$ in (1) and (2),

$$x^2 - vx^2 + v^2x^2 = 19; \text{ or, } x^2 = \frac{19}{1 - v + v^2}$$

$$2x^2 - v^2x^2 = -17; \text{ or, } x^2 = \frac{-17}{2 - v^2} \qquad (3)$$

Equating the values of x^2, $$\frac{19}{1 - v + v^2} = \frac{-17}{2 - v^2}$$

$$38 - 19v^2 = -17 + 17v - 17v^2$$

$$-2v^2 - 17v = -55$$

$$16v^2 + 136v + 289 = 729$$

$$4v + 17 = \pm 27$$

$$4v = 10 \text{ or } -44$$

$$v = \frac{5}{2} \text{ or } -11$$

Substituting in (3), $$x^2 = \frac{-17}{2 - \frac{25}{4}} \text{ or } \frac{-17}{2 - 121}$$

$$= 4 \text{ or } \frac{1}{7}$$

$$x = \pm 2 \text{ or } \pm \frac{1}{\sqrt{7}}$$

$$= \pm 2 \text{ or } \pm \frac{1}{7}\sqrt{7}$$

Substituting these values in $y = vx$, $$y = \frac{5}{2}(\pm 2) \text{ or } -11\left(\pm \frac{1}{7}\sqrt{7}\right)$$

$$= \pm 5 \text{ or } \mp \frac{11}{7}\sqrt{7}$$

31. $$\begin{cases} x + y = 3(a - b) & (1) \\ xy = 2a^2 - 5ab + 2b^2 & (2) \end{cases}$$

Squaring (1), $$x^2 + 2xy + y^2 = 9a^2 - 18ab + 9b^2 \qquad (3)$$

From (2), $$4xy = 8a^2 - 20ab + 8b^2 \qquad (4)$$

Subtracting (4) from (3), $$x^2 - 2xy + y^2 = a^2 + 2ab + b^2$$

$$x - y = \pm (a + b) \qquad (5)$$

Adding (1) and (5), $$2x = 3(a - b) \pm (a + b)$$

$$= 4a - 2b \text{ or } 2a - 4b$$

$$x = 2a - b \text{ or } a - 2b$$

Subtracting (5) from (1), $$2y = 3(a - b) \mp (a + b)$$

$$= 2a - 4b \text{ or } 4a - 2b$$

$$y = a - 2b \text{ or } 2a - b$$

32.
$$\begin{cases} x^5 + y^5 = 33 & (1) \\ x + y = 3 & (2) \end{cases}$$

Putting $x = u + v$ and $y = u - v$,
$$(u+v)^5 + (u-v)^5 = 33$$
$$u + v + u - v = 3$$

Reducing (2), $$2u = 3$$
$$u = \frac{3}{2}$$

Reducing (1), $$2u^5 + 20u^3v^2 + 10uv^4 = 33$$

Substituting the value of u,
$$\frac{243}{16} + \frac{135}{2}v^2 + 15v^4 = 33$$
$$240v^4 + 1080v^2 = 285$$
$$16v^4 + 72v^2 = 19$$
$$16v^4 + 72v^2 + 81 = 100$$
$$4v^2 + 9 = \pm 10$$
$$4v^2 = 1 \text{ or } -19$$
$$v^2 = \frac{1}{4} \text{ or } -\frac{19}{4}$$
$$v = \pm\frac{1}{2} \text{ or } \pm\frac{1}{2}\sqrt{-19}$$

Therefore, $$x = u + v = \frac{3}{2} \pm \frac{1}{2} \text{ or } \frac{3}{2} \pm \frac{1}{2}\sqrt{-19}$$
$$= 2, 1, \text{ or } \frac{3 \pm \sqrt{-19}}{2}$$
$$y = u - v = \frac{3}{2} \mp \frac{1}{2} \text{ or } \frac{3}{2} \mp \frac{1}{2}\sqrt{-19}$$
$$= 1, 2, \text{ or } \frac{3 \mp \sqrt{-19}}{2}$$

33.
$$\begin{cases} xy^2 + y = 1 & (1) \\ x^2y^4 + y^2 = 5 & (2) \end{cases}$$

Squaring (1), $$x^2y^4 + 2xy^3 + y^2 = 1 \quad (3)$$

Subtracting (3) from (2), $$-2xy^3 = 4 \quad (4)$$

Adding (2) and (4), $$x^2y^4 - 2xy^3 + y^2 = 9$$
$$xy^2 - y = \pm 3 \quad (5)$$

Subtracting (5) from (1), $$2y = 1 \mp 3 = -2 \text{ or } 4$$
$$y = -1 \text{ or } 2$$

Substituting $y = -1$ in (1), $$x - 1 = 1$$
$$x = 2$$

Substituting $y = 2$ in (1), $$4x + 2 = 1$$
$$4x = -1$$
$$x = -\frac{1}{4}$$

34. $$\begin{cases} x+z=7 & (1)\\ 2y-3z=-5 & (2)\\ x^2+y^2-z^2=11 & (3)\end{cases}$$

From (1), $$x=7-z \qquad (4)$$

From (2), $$2y=3z-5$$

$$y=\frac{3z-5}{2} \qquad (5)$$

Substituting from (4) and (5) in (3),

$$(7-z)^2+\frac{(3z-5)^2}{4}-z^2=11$$

$$49-14z+z^2+\frac{(3z-5)^2}{4}-z^2=11$$

$$\frac{(3z-5)^2}{4}=14z-38$$

$$9z^2-30z+25=56z-152$$

$$9z^2-86z=-177$$

$$81z^2-774z+1849=256$$

$$9z-43=\pm 16$$

$$9z=27 \text{ or } 59$$

$$z=3 \text{ or } \frac{59}{9}$$

Substituting in (4) and (5), $$x=7-3 \text{ or } 7-\frac{59}{9}$$

$$=4 \text{ or } \frac{4}{9}$$

$$y=\frac{9-5}{2} \text{ or } \frac{\frac{59}{3}-5}{2}$$

$$=2 \text{ or } \frac{22}{3}$$

35. $$\begin{cases} 2x^2-7xy-2y^2=5 & (1)\\ 3xy-x^2+6y^2=44 & (2)\end{cases}$$

Putting $y=vx$ in (1) and (2),

$$2x^2-7vx^2-2v^2x^2=5;\text{ or, } x^2=\frac{5}{2-7v-2v^2} \qquad (3)$$

$$3vx^2-x^2+6v^2x^2=44;\text{ or, } x^2=\frac{44}{3v-1+6v^2}$$

Equating the values of x^2, $$\frac{5}{2-7v-2v^2}=\frac{44}{3v-1+6v^2}$$

$$15v-5+30v^2=88-308v-88v^2$$

$$118v^2+323v=93$$

$$4 \times 118^2 \cdot v^2 + 472 \times 323 v = 43896$$
$$236^2 v^2 + 472 \times 323 v + 323^2 = 148225$$
$$236 v + 323 = \pm 385$$
$$236 v = -708 \text{ or } 62$$
$$v = -3 \text{ or } \frac{31}{118}$$

Substituting in (3), $x^2 = \dfrac{5}{2+21-18} \text{ or } \dfrac{5}{2 - \frac{217}{118} - \frac{961}{6962}}$

$$= 1 \text{ or } \frac{3481}{16}$$
$$x = \pm 1 \text{ or } \pm \frac{59}{4}$$

Substituting these values in $y = vx$, $y = -3(\pm 1) \text{ or } \dfrac{31}{118}\left(\pm \dfrac{59}{4}\right)$

$$= \mp 3 \text{ or } \pm \frac{31}{8}$$

36. $\begin{cases} \dfrac{x-y}{x+y} - \dfrac{x+y}{x-y} = \dfrac{3}{2} & (1) \\ 2x - y = 7 & (2) \end{cases}$

From (1), $2(x-y)^2 - 2(x+y)^2 = 3(x^2 - y^2)$

$$2x^2 - 4xy + 2y^2 - 2x^2 - 4xy - 2y^2 = 3x^2 - 3y^2$$
$$3y^2 - 8xy - 3x^2 = 0 \qquad (3)$$

From (2), $y = 2x - 7 \qquad (4)$

Substituting in (3),

$$3(2x-7)^2 - 8x(2x-7) - 3x^2 = 0$$
$$12x^2 - 84x + 147 - 16x^2 + 56x - 3x^2 = 0$$
$$-7x^2 - 28x = -147$$
$$x^2 + 4x = 21$$
$$x^2 + 4x + 4 = 25$$
$$x + 2 = \pm 5$$
$$x = 3 \text{ or } -7$$

Substituting in (4), $y = 6 - 7 \text{ or } -14 - 7$

$$= -1 \text{ or } -21$$

37. $\begin{cases} x^2 + y^2 = 7 + xy \\ x^3 + y^3 = 6xy - 1 \end{cases}$

Putting $x = u + v$ and $y = u - v$,

$$(u+v)^2 + (u-v)^2 = 7 + u^2 - v^2 \qquad (1)$$
$$(u+v)^3 + (u-v)^3 = 6(u^2 - v^2) - 1 \qquad (2)$$

Reducing (1),
$$2u^2+2v^2=7+u^2-v^2$$
$$u^2+3v^2=7 \qquad (3)$$
$$3v^2=7-u^2 \qquad (4)$$

Reducing (2),
$$2u^3+6uv^2=6(u^2-v^2)-1$$
$$2u(u^2+3v^2)=6u^2-6v^2-1$$

Substituting from (3) and (4),
$$14u=6u^2-14+2u^2-1$$
$$-8u^2+14u=-15$$
$$64u^2-112u+49=169$$
$$8u-7=\pm 13$$
$$8u=20 \text{ or } -6$$
$$u=\frac{5}{2} \text{ or } -\frac{3}{4}$$

Substituting in (4),
$$3v^2=7-\frac{25}{4} \text{ or } 7-\frac{9}{16}$$
$$=\frac{3}{4} \text{ or } \frac{103}{16}$$
$$v^2=\frac{1}{4} \text{ or } \frac{309}{144}$$
$$v=\pm\frac{1}{2} \text{ or } \pm\frac{1}{12}\sqrt{309}$$

Therefore,
$$x=u+v=\frac{5}{2}\pm\frac{1}{2} \text{ or } -\frac{3}{4}\pm\frac{1}{12}\sqrt{309}$$
$$=3,\ 2, \text{ or } \frac{-9\pm\sqrt{309}}{12}$$

and
$$y=u-v=\frac{5}{2}\mp\frac{1}{2} \text{ or } -\frac{3}{4}\mp\frac{1}{12}\sqrt{309}$$
$$=2,\ 3, \text{ or } \frac{-9\mp\sqrt{309}}{12}$$

Art. 278. — Pages 243–245.

1. Let $x=$ the greater number,
and $y=$ the less.

By the conditions,
$$\begin{cases} x^2+y^2=106 & (1)\\ x^2-y^2=\frac{7}{2}(x-y)^2 & (2)\end{cases}$$

Dividing (2) by $x-y$,
$$x+y=\frac{7}{2}(x-y)$$
$$2x+2y=7x-7y$$
$$9y=5x$$
$$y=\frac{5x}{9} \qquad (3)$$

Substituting in (1), $x^2 + \frac{25x^2}{81} = 106$

$\frac{106x^2}{81} = 106$

$\frac{x^2}{81} = 1$

Whence, $x = 9$, the greater number.
Substituting in (3), $y = 5$, the less number.

2. Let $x =$ the greater number,
and $y =$ the less.

By the conditions, $\begin{cases} (x-y)y = 42 & (1) \\ (x-y)(x+y) = 133 & (2) \end{cases}$

Dividing (1) by (2), $\frac{y}{x+y} = \frac{6}{19}$

$19y = 6x + 6y$

$13y = 6x$

$x = \frac{13y}{6}$ (3)

Substituting in (1), $\left(\frac{13y}{6} - y\right)y = 42$

$\frac{7y^2}{6} = 42$

$\frac{y^2}{6} = 6$

Whence, $y = 6$, the less number.
Substituting in (3), $x = 13$, the greater number.

3. Let $x =$ the side of one field in rods,
and $y =$ the side of the other.

By the conditions, $\begin{cases} x^2 + y^2 = 1300 & (1) \\ 4x + 4y = 200 & (2) \end{cases}$

From (2), $x + y = 50$ (3)
Squaring (3), $x^2 + 2xy + y^2 = 2500$ (4)
Subtracting (1) from (4), $2xy = 1200$ (5)
Subtracting (5) from (1), $x^2 - 2xy + y^2 = 100$

$x - y = \pm 10$ (6)

Adding (3) and (6), $2x = 60 \text{ or } 40$

$x = 30 \text{ or } 20$

Subtracting (6) from (3), $2y = 40 \text{ or } 60$

$y = 20 \text{ or } 30$

Therefore the area of one field is 900 square rods, and of the other 400 square rods.

4. Let $x =$ the greater number,
and $y =$ the less.

By the conditions,
$$\begin{cases} x^2 - y^2 = 7 & (1) \\ x^2y^2 = 144 & (2) \end{cases}$$

Squaring (1), $x^4 - 2x^2y^2 + y^4 = 49$ (3)

From (2), $4x^2y^2 = 576$ (4)

Adding (3) and (4), $x^4 + 2x^2y^2 + y^4 = 625$

$x^2 + y^2 = 25$ (5)

Adding (1) and (5), $2x^2 = 32$

$x = 4$

Subtracting (1) from (5), $2y^2 = 18$

$y = 3$

5. Let $x =$ the length in rods,
and $y =$ the breadth.

By the conditions,
$$\begin{cases} (x+2)(y+3) = 108 & (1) \\ (x-2)(y-3) = 24 & (2) \end{cases}$$

From (1), $xy + 3x + 2y = 102$ (3)

From (2), $xy - 3x - 2y = 18$ (4)

Adding (3) and (4), $2xy = 120$ (5)

Subtracting (4) from (3), $6x + 4y = 84$

$3x + 2y = 42$

$$y = \frac{42 - 3x}{2} \quad (6)$$

Substituting in (5), $x(42 - 3x) = 120$

$14x - x^2 = 40$

$x^2 - 14x + 49 = -40 + 49 = 9$

$x - 7 = \pm 3$

$x = 10$ or 4

Substituting in (6), $y = 6$ or 15

The solution $x = 4$, $y = 15$ is inadmissible; therefore the length of the field is 10 rods, and its breadth is 6 rods.

6. Let x and y be the numbers; then by the conditions,
$$\begin{cases} x^3 + y^3 = 407 & (1) \\ x^2 + y^2 = xy + 37 & (2) \end{cases}$$

From (2), $x^2 - xy + y^2 = 37$ (3)

Dividing (1) by (3), $x + y = 11$ (4)

Squaring (4), $x^2 + 2xy + y^2 = 121$ (5)

Subtracting (5) from (3), $-3xy = -84$

$-xy = -28$ (6)

Adding (3) and (6), $x^2 - 2xy + y^2 = 9$

$x - y = \pm 3$ (7)

Adding (4) and (7), $2x = 14$ or 8

$x = 7$ or 4

Subtracting (7) from (4), $2y = 8$ or 14

$y = 4$ or 7

Therefore the numbers are 7 and 4.

7. Let $x =$ the price of a duck,

and $y =$ the price of a turkey.

By the conditions,
$$\begin{cases} 6x + 2y = 15 & (1) \\ \dfrac{14}{x} = \dfrac{9}{y} + 4 & (2) \end{cases}$$

From (2), $$14y = 9x + 4xy \quad (3)$$

From (1), $$y = \frac{15 - 6x}{2} \quad (4)$$

Substituting in (3),
$$105 - 42x = 9x + 30x - 12x^2$$
$$12x^2 - 81x = -105$$
$$4x^2 - 27x = -35$$
$$64x^2 - 432x + 729 = -560 + 729 = 169$$
$$8x - 27 = \pm 13$$
$$8x = 14 \text{ or } 40$$
$$x = \frac{7}{4} \text{ or } 5$$

Substituting in (4),
$$y = \frac{15 - \frac{21}{2}}{2} \text{ or } \frac{15 - 30}{2}$$
$$= \frac{9}{4} \text{ or } -\frac{15}{2}$$

The solution $x = 5$, $y = -\dfrac{15}{2}$ is inadmissible; therefore the price of a duck is \$1.75, and of a turkey \$2.25.

8. Let $x =$ the first digit,

and $y =$ the second.

Then, $10x + y =$ the number,

and $10y + x =$ the no. with its digits inverted.

By the conditions,
$$\begin{cases} 10y + x + 10x + y = 33 & (1) \\ (10y + x)(10x + y) = 252 & (2) \end{cases}$$

From (1),
$$11x + 11y = 33$$
$$x + y = 3$$
$$y = 3 - x \quad (3)$$

From (2), $$10x^2 + 10y^2 + 101xy = 252$$

Substituting from (3),

$$10x^2 + 10(3-x)^2 + 101x(3-x) = 252$$
$$10x^2 + 90 - 60x + 10x^2 + 303x - 101x^2 = 252$$
$$-81x^2 + 243x = 162$$
$$x^2 - 3x = -2$$
$$4x^2 - 12x + 9 = -8 + 9 = 1$$
$$2x - 3 = \pm 1$$
$$2x = 4 \text{ or } 2$$
$$x = 2 \text{ or } 1$$

Substituting in (3), $y = 1$ or 2

Therefore the number is 21 or 12.

9. Let $x =$ the numerator,
and $y =$ the denominator.

By the conditions,
$$\begin{cases} x + y = 8 & (1) \\ \dfrac{x}{y}\left(\dfrac{x+1}{y+1}\right) = \dfrac{2}{5} & (2) \end{cases}$$

From (2), $5x^2 + 5x = 2y^2 + 2y$ (3)

From (1), $y = 8 - x$ (4)

Substituting in (3), $5x^2 + 5x = 2(8-x)^2 + 2(8-x)$

$$5x^2 + 5x = 128 - 32x + 2x^2 + 16 - 2x$$
$$3x^2 + 39x = 144$$

Multiplying by $\frac{4}{3}$, $4x^2 + 52x = 192$

$$4x^2 + 52x + 169 = 361$$
$$2x + 13 = \pm 19$$
$$2x = 6 \text{ or } -32$$
$$x = 3 \text{ or } -16$$

Substituting in (4), $y = 5$ or 24

Therefore the fraction is $\frac{3}{5}$ or $\frac{-16}{24}$.

10. Let $x =$ the length of the garden in feet,
and $y =$ its breadth.

By the conditions,
$$\begin{cases} xy = 15000 & (1) \\ 14x + 14y + 196 = 3696 & (2) \end{cases}$$

From (2), $x + y + 14 = 264$

$$x + y = 250 \quad (3)$$

Squaring (3), $x^2 + 2xy + y^2 = 62500$ (4)

From (1), $4xy = 60000$ (5)

Subtracting (5) from (4),

$$x^2 - 2xy + y^2 = 2500$$
$$x - y = \pm 50 \quad (6)$$

Adding (3) and (6), $2x = 300$ or 200

$x = 150$ or 100

Subtracting (6) from (3), $2y = 200$ or 300

$y = 100$ or 150

The solution $x = 100$, $y = 150$ is inadmissible; therefore the length of the garden is 150 feet, and its breadth is 100 feet.

11. Let $x =$ the length of the field in rods,

and $y =$ its breadth.

Then, $xy =$ its area.

By the conditions, $\begin{cases} xy = 160 & (1) \\ (x+4)(y+3) = xy + 100 & (2) \end{cases}$

From (2), $xy + 3x + 4y + 12 = xy + 100$

$$3x + 4y = 88$$

$$x = \frac{88 - 4y}{3} \qquad (3)$$

Substituting in (1), $y\left(\dfrac{88 - 4y}{3}\right) = 160$

$$88y - 4y^2 = 480$$

$$y^2 - 22y = -120$$

$$y^2 - 22y + 121 = 1$$

$$y - 11 = \pm 1$$

$$y = 10 \text{ or } 12$$

Substituting in (3), $x = 16$ or $\dfrac{40}{3}$

Therefore the length of the field is 16 rods, and its breadth 10 rods; or, the length is $13\frac{1}{3}$ rods, and the breadth 12 rods.

12. Let $x =$ the rate of the boatman,

and $y =$ the rate of the stream.

By the conditions, $\begin{cases} \dfrac{12}{x+y} = \dfrac{12}{x-y} - 4 & (1) \\ 2x + y = 10 & (2) \end{cases}$

From (1), $\dfrac{3}{x+y} = \dfrac{3}{x-y} - 1$

$$3x - 3y = 3x + 3y - x^2 + y^2$$

Or, $x^2 = y^2 + 6y \qquad (3)$

From (2), $x = \dfrac{10 - y}{2} \qquad (4)$

Substituting in (3), $\dfrac{(10 - y)^2}{4} = y^2 + 6y$

$$100 - 20y + y^2 = 4y^2 + 24y$$

$$3y^2 + 44y = 100$$

$$9y^2 + 132y + 484 = 300 + 484 = 784$$
$$3y + 22 = \pm 28$$
$$3y = 6$$

Whence, $y = 2$, the rate of the stream.

Substituting in (4), $x = 4$, the rate of the boatman.

13. Let $x =$ the no. of acres A receives,

and $y =$ the no. B receives.

By the conditions,
$$\begin{cases} x + y = 104 & (1) \\ \dfrac{320}{x} = \dfrac{320}{y} + 3 & (2) \end{cases}$$

From (2), $320y = 320x + 3xy$ (3)

From (1), $y = 104 - x$ (4)

Substituting in (3), $33280 - 320x = 320x + 312x - 3x^2$
$$3x^2 - 952x = -33280$$
$$9x^2 - 2856x + 476^2 = -99840 + 226576 = 126736$$
$$3x - 476 = \pm 356$$
$$3x = 120$$

Whence, $x = 40$, the no. of acres A receives.

Substituting in (4), $y = 64$, the no. of acres B receives.

Therefore, $\dfrac{\$320}{40} = \$8 =$ A's price per acre,

and $\dfrac{\$320}{64} = \$5 =$ B's price per acre.

14. Let $x + y$ and $x - y$ represent the numbers; then by the conditions,
$$\begin{cases} x^2 - y^2 + 2x = 47 & (1) \\ (x+y)^2 + (x-y)^2 = 2x + 62 & (2) \end{cases}$$

From (2), $2x^2 + 2y^2 = 2x + 62$

Or, $x^2 + y^2 - x = 31$ (3)

Adding (1) and (3), $2x^2 + x = 78$
$$16x^2 + 8x + 1 = 624 + 1 = 625$$
$$4x + 1 = \pm 25$$
$$4x = 24$$
$$x = 6$$

Substituting in (3), $36 + y^2 - 6 = 31$
$$y^2 = 1$$
$$y = 1$$

Therefore the numbers are $6 + 1$ and $6 - 1$; that is, 7 and 5.

15. Let $x + y$ and $x - y$ represent the numbers; then by the conditions,
$$\begin{cases} 2x = 7 & (1) \\ (x+y)^4 + (x-y)^4 = 641 & (2) \end{cases}$$

From (1), $x = \frac{7}{2}$

From (2), $2x^4 + 12x^2y^2 + 2y^4 = 641$

Putting $x = \frac{7}{2}$, $\frac{2401}{8} + 147y^2 + 2y^4 = 641$

$$16y^4 + 1176y^2 + 2401 = 5128$$

$$16y^4 + 1176y^2 = 2727$$

Completing the square,

$$16y^4 + 1176y^2 + 147^2 = 2727 + 21609$$

$$= 24336$$

$$4y^2 + 147 = \pm 156$$

$$4y^2 = 9$$

$$y = \frac{3}{2}$$

Therefore the numbers are $\frac{7}{2} + \frac{3}{2}$ and $\frac{7}{2} - \frac{3}{2}$; that is, 5 and 2.

16. Let $x =$ the circumference of the hind-wheel in yds.,
and $y =$ the circumference of the fore-wheel.

By the conditions,

$$\begin{cases} \dfrac{180}{x} = \dfrac{180}{y} - 15 & (1) \\ \dfrac{180}{x+1} = \dfrac{180}{y+1} - 9 & (2) \end{cases}$$

From (1), $\frac{12}{x} = \frac{12}{y} - 1$

$$12y = 12x - xy \qquad (3)$$

From (2), $\frac{20}{x+1} = \frac{20}{y+1} - 1$

$$20y + 20 = 20x + 20 - xy - x - y - 1$$

$$21y = 19x - xy - 1 \qquad (4)$$

Subtracting (3) from (4),

$$9y = 7x - 1$$

$$y = \frac{7x-1}{9} \qquad (5)$$

Substituting in (3),

$$\frac{12(7x-1)}{9} = 12x - x\left(\frac{7x-1}{9}\right)$$

$$84x - 12 = 108x - 7x^2 + x$$

$$7x^2 - 25x = 12$$

$$196x^2 - 700x + 625 = 336 + 625 = 961$$
$$14x - 25 = \pm 31$$
$$14x = 56$$

Whence, $x = 4$, the circumference of the hind-wheel in yds.
Substituting in (5), $y = 3$, the circumference of the fore-wheel.

17. Let $x =$ the first rate,
and $y =$ the second.

Then, $\dfrac{36}{\frac{y}{100}}$, or $\dfrac{3600}{y} =$ the first portion,

and $\dfrac{49}{\frac{x}{100}}$, or $\dfrac{4900}{x} =$ the second portion.

By the conditions,
$$\begin{cases} \dfrac{3600}{x} + \dfrac{4900}{y} = 1300 & (1) \\ \dfrac{3600}{y} \times \dfrac{x}{100} = \dfrac{4900}{x} \times \dfrac{y}{100} & (2) \end{cases}$$

From (1), $36x + 49y = 13xy$ (3)

From (2), $36x^2 = 49y^2$

or, $y = \dfrac{6x}{7}$ (4)

Substituting from (4) in (3),
$$36x + 42x = \frac{78x^2}{7}$$
$$1 = \frac{x}{7}$$

Whence, $x = 7$, the first rate.
Substituting in (4), $y = 6$, the second rate.

18. Let $x =$ the length in yards,
and $y =$ the width.

After shrinkage the length is $\dfrac{7x}{8}$ yards, and the width is $\dfrac{15y}{16}$ yards.

Hence the diminution in the surface is $xy - \dfrac{7x}{8} \times \dfrac{15y}{16}$, or $\dfrac{23xy}{128}$,

and the diminution in the sum of the four sides is $\dfrac{2x}{8} + \dfrac{2y}{16}$, or $\dfrac{x}{4} + \dfrac{y}{8}$.

Then by the conditions,

$$\begin{cases} \dfrac{23\,xy}{128} = \dfrac{23}{4} & (1) \\[2ex] \dfrac{x}{4} + \dfrac{y}{8} = \dfrac{17}{4} & (2) \end{cases}$$

From (1), $xy = 32$ (3)

From (2), $2x + y = 34$

$$y = 34 - 2x$$

Substituting in (3), $x(34 - 2x) = 32$

$$2x^2 - 34x = -32$$

$$4x^2 - 68x + 289 = -64 + 289 = 225$$

$$2x - 17 = \pm 15$$

$$2x = 32 \text{ or } 2$$

$$x = 16 \text{ or } 1$$

Substituting in (3), $y = 2 \text{ or } 32$

The solution $x = 1$, $y = 32$ is inadmissible; therefore the length of the cloth was 16 yards, and its width 2 yards.

CHAPTER XXV.

Art. 281. — Page 247.

1. $x^2 + 5x = -2$

Sum of roots $= -5$

Product of roots $= 2$

2. $x^2 - 7x = -11$

Sum of roots $= 7$

Product of roots $= 11$

3. $x^2 + 6x = 1$

Sum of roots $= -6$

Product of roots $= -1$

4. $x^2 - \frac{3x}{2} = 1$

Sum of roots $= \frac{3}{2}$

Product of roots $= -1$

5. $x^2 - \frac{x}{8} = -\frac{1}{2}$

Sum of roots $= \frac{1}{8}$

Product of roots $= \frac{1}{2}$

6. $4x^2 - 6x = 3$

$$x^2 - \frac{3x}{2} = \frac{3}{4}$$

Sum of roots $= \frac{3}{2}$

Product of roots $= -\frac{3}{4}$

7. $14x^2 + 12x = 7$

$$x^2 + \frac{6x}{7} = \frac{1}{2}$$

Sum of roots $= -\frac{6}{7}$

Product of roots $= -\frac{1}{2}$

8. $x^2 - ax = -\frac{a^2 - b^2}{4}$

Sum of roots $= a$

Product of roots $= \frac{a^2 - b^2}{4}$

9. $(x-4)(x-5) = 0$

$$x^2 - 9x + 20 = 0$$

$$x^2 - 9x = -20$$

10. $(x-1)(x+3) = 0$

$$x^2 + 2x - 3 = 0$$

$$x^2 + 2x = 3$$

11. $(x-3)\left(x + \frac{3}{5}\right) = 0$

$$(x-3)(5x+3) = 0$$

$$5x^2 - 12x - 9 = 0$$

$$5x^2 - 12x = 9$$

12. $(x-7)\left(x + \frac{19}{3}\right) = 0$

$$(x-7)(3x+19) = 0$$

$$3x^2 - 2x - 133 = 0$$

$$3x^2 - 2x = 133$$

13.
$$\begin{aligned} \left(x-\frac{2}{3}\right)\left(x-\frac{3}{4}\right)&=0\\ (3x-2)(4x-3)&=0\\ 12x^2-17x+6&=0\\ 12x^2-17x&=-6 \end{aligned}$$

14.
$$\begin{aligned} \left(x+\frac{8}{3}\right)\left(x-\frac{4}{7}\right)&=0\\ (3x+8)(7x-4)&=0\\ 21x^2+44x-32&=0\\ 21x^2+44x&=32 \end{aligned}$$

15.
$$\begin{aligned} \left(x+\frac{5}{3}\right)\left(x+\frac{7}{2}\right)&=0\\ (3x+5)(2x+7)&=0\\ 6x^2+31x+35&=0\\ 6x^2+31x&=-35 \end{aligned}$$

16.
$$\begin{aligned} \left(x+\frac{17}{3}\right)x&=0\\ (3x+17)x&=0\\ 3x^2+17x&=0 \end{aligned}$$

17.
$$\begin{aligned} (x-a+b)(x-a-2b)&=0\\ x^2-2ax-bx+a^2+ab-2b^2&=0\\ x^2-2ax-bx&=-a^2-ab+2b^2 \end{aligned}$$

18.
$$\begin{aligned} [x-m(1+m)][x-m(1-m)]&=0\\ (x-m-m^2)(x-m+m^2)&=0\\ (x-m)^2-m^4&=0\\ x^2-2mx+m^2-m^4&=0\\ x^2-2mx&=m^4-m^2 \end{aligned}$$

19.
$$\begin{aligned} (x-2-\sqrt{3})(x-2+\sqrt{3})&=0\\ (x-2)^2-3&=0\\ x^2-4x+4-3&=0\\ x^2-4x&=-1 \end{aligned}$$

20.
$$\begin{aligned} \left(x-\frac{m+\sqrt{n}}{2}\right)\left(x-\frac{m-\sqrt{n}}{2}\right)&=0\\ (2x-m-\sqrt{n})(2x-m+\sqrt{n})&=0\\ (2x-m)^2-n&=0\\ 4x^2-4mx+m^2-n&=0\\ 4x^2-4mx&=n-m^2 \end{aligned}$$

Art. 282. — Page 249.

5.
$$\left(x-\frac{3}{5}\right)\left(x+\frac{7}{2}\right)=0$$
$$x-\frac{3}{5}=0, \text{ or } x=\frac{3}{5}$$
$$x+\frac{7}{2}=0, \text{ or } x=-\frac{7}{2}$$

6.
$$\begin{aligned} 2x^2-x&=0\\ x(2x-1)&=0\\ x&=0 \end{aligned}$$
$$2x-1=0, \text{ or } x=\frac{1}{2}$$

7. $(ax+b)(bx-a)=0$

$ax+b=0$, or $x=-\dfrac{b}{a}$

$bx-a=0$, or $x=\dfrac{a}{b}$

8. $(x^2-4)(x^2-9)=0$

$x^2-4=0$, or $x=\pm 2$

$x^2-9=0$, or $x=\pm 3$

9. $(x-2)(x^2+9x+20)=0$

$(x-2)(x+4)(x+5)=0$

$x-2=0$, or $x=2$

$x+4=0$, or $x=-4$

$x+5=0$, or $x=-5$

Hence, $x=2, -4$, or -5

10. $2x^3-18x=0$

$2x(x^2-9)=0$

$x=0$

$x^2-9=0$, or $x=\pm 3$

11. $(3x+1)(4x^2-25)=0$

$3x+1=0$, or $x=-\dfrac{1}{3}$

$4x^2-25=0$, or $x=\pm\dfrac{5}{2}$

12. $3x^3+12x^2=0$

$3x^2(x+4)=0$

$x^2=0$, or $x=0$

$x+4=0$, or $x=-4$

13. $(x^2-a^2)(x^2-ax-b)=0$

$x^2-a^2=0$, or $x=\pm a$

$x^2-ax-b=0$

$4x^2-4ax=4b$

$4x^2-4ax+a^2=a^2+4b$

$2x-a=\pm\sqrt{a^2+4b}$

$2x=a\pm\sqrt{a^2+4b}$

$x=\dfrac{a\pm\sqrt{a^2+4b}}{2}$

Hence, $x=\pm a$ or $\dfrac{a\pm\sqrt{a^2+4b}}{2}$

14. $24x^3-2x^2-12x=0$

$2x(12x^2-x-6)=0$

$x=0$

$12x^2-x-6=0$

$576x^2-48x+1=288+1=289$

$24x-1=\pm 17$

$24x=18$ or -16

$x=\dfrac{3}{4}$ or $-\dfrac{2}{3}$

Hence, $x=0, \dfrac{3}{4}$, or $-\dfrac{2}{3}$

15.
$$x(2x+5)(3x-7)(4x+1)=0$$
$$x=0$$
$$2x+5=0,\text{ or } x=-\frac{5}{2}$$
$$3x-7=0,\text{ or } x=\frac{7}{3}$$
$$4x+1=0,\text{ or } x=-\frac{1}{4}$$

16. $(x^2-5x+6)(x^2+7x+12)(x^2-3x-4)=0$
$$(x-2)(x-3)(x+3)(x+4)(x-4)(x+1)=0$$
$$x-2=0,\text{ or } x=2$$
$$x-3=0,\text{ or } x=3$$
$$x+3=0,\text{ or } x=-3$$
$$x+4=0,\text{ or } x=-4$$
$$x-4=0,\text{ or } x=4$$
$$x+1=0,\text{ or } x=-1$$
Hence, $x=-1,\ 2,\ \pm 3,\text{ or } \pm 4$

17.
$$x^3+1=0$$
$$(x+1)(x^2-x+1)=0$$
$$x+1=0,\text{ or } x=-1$$
$$x^2-x+1=0$$
$$4x^2-4x=-4$$
$$4x^2-4x+1=-3$$
$$2x-1=\pm\sqrt{-3}$$
$$x=\frac{1\pm\sqrt{-3}}{2}$$
Hence, $x=-1\text{ or }\frac{1\pm\sqrt{-3}}{2}$

18.
$$x^6-1=0$$
$$(x^3+1)(x^3-1)=0$$
$$(x+1)(x^2-x+1)(x-1)(x^2+x+1)=0$$
$$x+1=0,\text{ or } x=-1$$
$$x^2-x+1=0,\text{ or } x=\frac{1\pm\sqrt{-3}}{2}$$
$$x-1=0,\text{ or } x=1$$
$$x^2+x+1=0,\text{ or } x=\frac{-1\pm\sqrt{-3}}{2}$$
Hence, $x=\pm 1\text{ or }\frac{\pm 1\pm\sqrt{-3}}{2}$

19.
$$x^3 - x^2 - 9x + 9 = 0$$
$$(x-1)(x^2-9) = 0$$
$$x - 1 = 0, \text{ or } x = 1$$
$$x^2 - 9 = 0, \text{ or } x = \pm 3$$

20.
$$2x^3 + 3x^2 - 2x - 3 = 0$$
$$(2x+3)(x^2-1) = 0$$
$$2x + 3 = 0, \text{ or } x = -\frac{3}{2}$$
$$x^2 - 1 = 0, \text{ or } x = \pm 1$$

Art. 283. — Page 251.

3.
$$x^2 + 13x + 40 = 0$$
Solving,
$$x = -5 \text{ or } -8$$
$$\therefore x^2 + 13x + 40 = (x+5)(x+8)$$

4.
$$x^2 - 11x + 18 = 0$$
Solving,
$$x = 2 \text{ or } 9$$
$$\therefore x^2 - 11x + 18 = (x-2)(x-9)$$

5.
$$x^2 - 4x - 60 = 0$$
Solving,
$$x = 10 \text{ or } -6$$
$$\therefore x^2 - 4x - 60 = (x-10)(x+6)$$

6.
$$2x^2 - 7x - 15 = 0$$
Solving,
$$x = 5 \text{ or } -\frac{3}{2}$$
$$\therefore 2x^2 - 7x - 15 = 2(x-5)\left(x+\frac{3}{2}\right)$$
$$= (x-5)(2x+3).$$

7.
$$4x^2 - 15x + 9 = 0$$
Solving,
$$x = 3 \text{ or } \frac{3}{4}$$
$$\therefore 4x^2 - 15x + 9 = 4(x-3)\left(x-\frac{3}{4}\right)$$
$$= (x-3)(4x-3)$$

8.
$$5x^2 + 36x + 7 = 0$$
Solving,
$$x = -\frac{1}{5} \text{ or } -7$$
$$\therefore 5x^2 + 36x + 7 = 5\left(x+\frac{1}{5}\right)(x+7)$$
$$= (5x+1)(x+7)$$

9.
$$4x^2+15x-4=0$$
Solving,
$$x=-4 \text{ or } \frac{1}{4}$$
$$\therefore 4x^2+15x-4=4(x+4)\left(x-\frac{1}{4}\right)$$
$$=(x+4)(4x-1)$$

10.
$$39-10x-x^2=0$$
Solving,
$$x=-13 \text{ or } 3$$
$$\therefore 39-10x-x^2=-(x+13)(x-3)$$
$$=(13+x)(3-x)$$

11.
$$2+x-6x^2=0$$
Solving,
$$x=\frac{2}{3} \text{ or } -\frac{1}{2}$$
$$\therefore 2+x-6x^2=-6\left(x-\frac{2}{3}\right)\left(x+\frac{1}{2}\right)$$
$$=-3\left(x-\frac{2}{3}\right)2\left(x+\frac{1}{2}\right)$$
$$=(2-3x)(1+2x)$$

12.
$$x^2-4x+1=0$$
Solving,
$$x=2\pm\sqrt{3}$$
$$\therefore x^2-4x+1=(x-2+\sqrt{3})(x-2-\sqrt{3})$$

13.
$$9x^2-6x-4=0$$
Solving,
$$x=\frac{1\pm\sqrt{5}}{3}$$
$$\therefore 9x^2-6x-4=9\left(x-\frac{1-\sqrt{5}}{3}\right)\left(x-\frac{1+\sqrt{5}}{3}\right)$$
$$=(3x-1+\sqrt{5})(3x-1-\sqrt{5})$$

14.
$$8x^2-18x+9=0$$
Solving,
$$x=\frac{3}{4} \text{ or } \frac{3}{2}$$
$$\therefore 8x^2-18x+9=8\left(x-\frac{3}{4}\right)\left(x-\frac{3}{2}\right)$$
$$=(4x-3)(2x-3)$$

15.
$$6-x-2x^2=0$$
Solving,
$$x=-2 \text{ or } \frac{3}{2}$$
$$\therefore\ 6-x-2x^2=-2(x+2)\left(x-\frac{3}{2}\right)$$
$$=(2+x)(3-2x)$$

16.
$$7x^2+50x+7=0$$
Solving,
$$x=-\frac{1}{7} \text{ or } -7$$
$$\therefore\ 7x^2+50x+7=7\left(x+\frac{1}{7}\right)(x+7)$$
$$=(7x+1)(x+7)$$

17.
$$6x^2-13ax-15a^2=0$$
Solving,
$$x=3a \text{ or } -\frac{5a}{6}$$
$$\therefore\ 6x^2-13ax-15a^2=6(x-3a)\left(x+\frac{5a}{6}\right)$$
$$=(x-3a)(6x+5a)$$

18.
$$5+4x-12x^2=0$$
Solving,
$$x=\frac{5}{6} \text{ or } -\frac{1}{2}$$
$$\therefore\ 5+4x-12x^2=-12\left(x-\frac{5}{6}\right)\left(x+\frac{1}{2}\right)$$
$$=-6\left(x-\frac{5}{6}\right)2\left(x+\frac{1}{2}\right)$$
$$=(5-6x)(1+2x)$$

19.
$$9x^2-12x+1=0$$
Solving,
$$x=\frac{2\pm\sqrt{3}}{3}$$
$$\therefore\ 9x^2-12x+1=9\left(x-\frac{2-\sqrt{3}}{3}\right)\left(x-\frac{2+\sqrt{3}}{3}\right)$$
$$=(3x-2+\sqrt{3})(3x-2-\sqrt{3})$$

20.
$$12x^2-7xy-10y^2=0$$
Solving,
$$x=-\frac{2y}{3} \text{ or } \frac{5y}{4}$$
$$\therefore\ 12x^2-7xy-10y^2=12\left(x+\frac{2y}{3}\right)\left(x-\frac{5y}{4}\right)$$
$$=(3x+2y)(4x-5y)$$

21. $$8x^2+18xy-5y^2=0$$

Solving, $$x=\frac{y}{4} \text{ or } -\frac{5y}{2}$$

$$\therefore 8x^2+18xy-5y^2=8\left(x-\frac{y}{4}\right)\left(x+\frac{5y}{2}\right)$$
$$=(4x-y)(2x+5y)$$

22. $$10x^2-23x+6=0$$

Solving, $$x=\frac{3}{10} \text{ or } 2$$

$$\therefore 10x^2-23x+6=10\left(x-\frac{3}{10}\right)(x-2)$$
$$=(10x-3)(x-2)$$

23. $$20x^2+41mx+20m^2=0$$

Solving, $$x=-\frac{5m}{4} \text{ or } -\frac{4m}{5}$$

$$\therefore 20x^2+41mx+20m^2=20\left(x+\frac{5m}{4}\right)\left(x+\frac{4m}{5}\right)$$
$$=(4x+5m)(5x+4m)$$

24. $$16x^2-34x+15=0$$

Solving, $$x=\frac{5}{8} \text{ or } \frac{3}{2}$$

$$\therefore 16x^2-34x+15=16\left(x-\frac{5}{8}\right)\left(x-\frac{3}{2}\right)$$
$$=(8x-5)(2x-3)$$

25. $$1-8x-x^2=0$$

Solving, $$x=-4\pm\sqrt{17}$$
$$\therefore 1-8x-x^2=-(x+4+\sqrt{17})(x+4-\sqrt{17})$$
$$=(\sqrt{17}+4+x)(\sqrt{17}-4-x)$$

26. $$15b^2+26bx-24x^2=0$$

Solving, $$x=\frac{3b}{2} \text{ or } -\frac{5b}{12}$$

$$\therefore 15b^2+26bx-24x^2=-24\left(x-\frac{3b}{2}\right)\left(x+\frac{5b}{12}\right)$$
$$=-2\left(x-\frac{3b}{2}\right)12\left(x+\frac{5b}{12}\right)$$
$$=(3b-2x)(5b+12x)$$

27. $$21x^2 + 58mnx + 21m^2n^2 = 0$$

Solving, $$x = -\frac{3mn}{7} \text{ or } -\frac{7mn}{3}$$

$$\therefore\ 21x^2 + 58mnx + 21m^2n^2 = 21\left(x + \frac{3mn}{7}\right)\left(x + \frac{7mn}{3}\right)$$
$$= (7x + 3mn)(3x + 7mn)$$

28. $$25x^2 - 20x - 2 = 0$$

Solving, $$x = \frac{2 \pm \sqrt{6}}{5}$$

$$\therefore\ 25x^2 - 20x - 2 = 25\left(x - \frac{2 - \sqrt{6}}{5}\right)\left(x - \frac{2 + \sqrt{6}}{5}\right)$$
$$= (5x - 2 + \sqrt{6})(5x - 2 - \sqrt{6})$$

Art. 284. — Page 252.

4. $$x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2$$
$$= (x^2 + 1)^2 - x^2$$
$$= (x^2 + x + 1)(x^2 - x + 1)$$

5. $$x^4 - 7x^2 + 1 = x^4 + 2x^2 + 1 - 9x^2$$
$$= (x^2 + 1)^2 - (3x)^2$$
$$= (x^2 + 3x + 1)(x^2 - 3x + 1)$$

6. $$4a^4 - 8a^2b^2 + b^4 = 4a^4 - 4a^2b^2 + b^4 - 4a^2b^2$$
$$= (2a^2 - b^2)^2 - (2ab)^2$$
$$= (2a^2 + 2ab - b^2)(2a^2 - 2ab - b^2)$$

7. $$m^4 - 14m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4 - 16m^2n^2$$
$$= (m^2 + n^2)^2 - (4mn)^2$$
$$= (m^2 + 4mn + n^2)(m^2 - 4mn + n^2)$$

8. $$1 - 13b^2 + 4b^4 = 1 - 4b^2 + 4b^4 - 9b^2$$
$$= (1 - 2b^2)^2 - (3b)^2$$
$$= (1 + 3b - 2b^2)(1 - 3b - 2b^2)$$

9. $$x^4 - 12x^2y^2 + 4y^4 = x^4 + 4x^2y^2 + 4y^4 - 16x^2y^2$$
$$= (x^2 + 2y^2)^2 - (4xy)^2$$
$$= (x^2 + 4xy + 2y^2)(x^2 - 4xy + 2y^2)$$

10.
$$\begin{aligned}4a^4 + 8a^2 + 9 &= 4a^4 + 12a^2 + 9 - 4a^2\\ &= (2a^2 + 3)^2 - (2a)^2\\ &= (2a^2 + 2a + 3)(2a^2 - 2a + 3)\end{aligned}$$

11.
$$\begin{aligned}4m^4 - 24m^2 + 25 &= 4m^4 - 20m^2 + 25 - 4m^2\\ &= (2m^2 - 5)^2 - (2m)^2\\ &= (2m^2 + 2m - 5)(2m^2 - 2m - 5)\end{aligned}$$

12.
$$\begin{aligned}a^4 - 5a^2x^2 + x^4 &= a^4 - 2a^2x^2 + x^4 - 3a^2x^2\\ &= (a^2 - x^2)^2 - (ax\sqrt{3})^2\\ &= (a^2 + ax\sqrt{3} - x^2)(a^2 - ax\sqrt{3} - x^2)\end{aligned}$$

13.
$$\begin{aligned}x^4 + 1 &= x^4 + 2x^2 + 1 - 2x^2\\ &= (x^2 + 1)^2 - (x\sqrt{2})^2\\ &= (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1)\end{aligned}$$

14.
$$\begin{aligned}4a^4 + 15a^2b^2 + 16b^4 &= 4a^4 + 16a^2b^2 + 16b^4 - a^2b^2\\ &= (2a^2 + 4b^2)^2 - (ab)^2\\ &= (2a^2 + ab + 4b^2)(2a^2 - ab + 4b^2)\end{aligned}$$

15.
$$\begin{aligned}16x^4 - 49m^2x^2 + 9m^4 &= 16x^4 - 24m^2x^2 + 9m^4 - 25m^2x^2\\ &= (4x^2 - 3m^2)^2 - (5mx)^2\\ &= (4x^2 + 5mx - 3m^2)(4x^2 - 5mx - 3m^2)\end{aligned}$$

16.
$$\begin{aligned}9x^4 - 6x^2 + 4 &= 9x^4 + 12x^2 + 4 - 18x^2\\ &= (3x^2 + 2)^2 - (3x\sqrt{2})^2\\ &= (3x^2 + 3x\sqrt{2} + 2)(3x^2 - 3x\sqrt{2} + 2)\end{aligned}$$

17.
$$\begin{aligned}9a^4 + 14a^2m^2 + 25m^4 &= 9a^4 + 30a^2m^2 + 25m^4 - 16a^2m^2\\ &= (3a^2 + 5m^2)^2 - (4am)^2\\ &= (3a^2 + 4am + 5m^2)(3a^2 - 4am + 5m^2)\end{aligned}$$

18.
$$\begin{aligned}4 - 32n^2 + 49n^4 &= 4 - 28n^2 + 49n^4 - 4n^2\\ &= (2 - 7n^2)^2 - (2n)^2\\ &= (2 + 2n - 7n^2)(2 - 2n - 7n^2)\end{aligned}$$

19.
$$\begin{aligned}16x^4 - 49x^2y^2 + 25y^4 &= 16x^4 - 40x^2y^2 + 25y^4 - 9x^2y^2\\ &= (4x^2 - 5y^2)^2 - (3xy)^2\\ &= (4x^2 + 3xy - 5y^2)(4x^2 - 3xy - 5y^2)\end{aligned}$$

Art. 286.—Page 254.

2. $x^2+2x-15=0$; or, $x^2+2x=15$.

Since q and p are both positive, the roots are one positive and the other negative; and the negative root is numerically the greater.

$p^2+4q=4+60=64$, a perfect square. Hence the roots are both rational.

3. $x^2+5x+6=0$; or, $x^2+5x=-6$.

Since q is negative, $4q$ numerically $<p^2$, and p positive, the roots are both negative.

$p^2+4q=25-24=1$, a perfect square. Hence the roots are both rational.

4. $x^2-10x=-25$.

Since q is negative, $4q$ numerically $=p^2$, and p negative, the roots are equal and positive.

$p^2+4q=100-100=0$. Hence the roots are both rational.

5. $3x^2-5x+4=0$; or, $x^2-\frac{5x}{3}=-\frac{4}{3}$.

Since q is negative and $4q$ numerically $>p^2$, the roots are imaginary.

6. $6x^2-7x-5=0$; or, $x^2-\frac{7x}{6}=\frac{5}{6}$.

Since q is positive and p negative, the roots are one positive and the other negative; and the positive root is numerically the greater.

$p^2+4q=\frac{49}{36}+\frac{20}{6}=\frac{169}{36}$, a perfect square. Hence the roots are both rational.

7. $9x^2+30x=-25$; or, $x^2+\frac{10x}{3}=-\frac{25}{9}$.

Since q is negative, $4q$ numerically $=p^2$, and p positive, the roots are equal and negative.

$p^2+4q=\frac{100}{9}-\frac{100}{9}=0$. Hence the roots are both rational.

8. $9x^2 + 8 = 18x$; or, $x^2 - 2x = -\frac{8}{9}$.

Since q is negative, $4q$ numerically $< p^2$, and p negative, the roots are both positive.

$p^2 + 4q = 4 - \frac{32}{9} = \frac{4}{9}$, a perfect square. Hence the roots are both rational.

9. $10 - 3x - 18x^2 = 0$; or, $x^2 + \frac{x}{6} = \frac{5}{9}$.

Since q and p are both positive, the roots are one positive and the other negative; and the negative root is numerically the greater.

$p^2 + 4q = \frac{1}{36} + \frac{20}{9} = \frac{81}{36}$, a perfect square. Hence the roots are both rational.

CHAPTER XXVI.

Art. 309.—Pages 262, 263.

3. Let $x =$ the first term.

$$\therefore x : 18 = 6 : 27$$

$$\frac{x}{18} = \frac{6}{27} = \frac{2}{9}$$

$$\therefore x = 4$$

4. Let $x =$ the second term.

$$\therefore 4 : x = 20 : 55$$

$$\frac{x}{4} = \frac{55}{20} = \frac{11}{4}$$

$$\therefore x = 11$$

5. Let $x =$ the fourth proportional.

$$\therefore \frac{2}{3} : \frac{3}{5} = \frac{2}{7} : x$$

Hence, by Art. 293,

$$\frac{2x}{3} = \frac{6}{35}$$

$$\therefore x = \frac{9}{35}$$

6. Let $x =$ the third proportional.

$$\therefore \frac{3}{4} : \frac{5}{6} = \frac{5}{6} : x$$

Hence, by Art. 293,

$$\frac{3x}{4} = \frac{25}{36}$$

$$\therefore x = \frac{25}{27}$$

7. By Art. 294, the mean proportional between 8 and 18 is

$$\sqrt{8 \times 18} = \sqrt{144} = 12.$$

8. The mean proportional between 14 and 42 is

$$\sqrt{14 \times 42} = \sqrt{14^2 \times 3} = 14\sqrt{3}.$$

9. The mean proportional between $2\frac{2}{9}$ and $\frac{5}{12}$ is

$$\sqrt{\frac{20}{9}\times\frac{5}{12}}=\sqrt{\frac{25}{27}}=\sqrt{\frac{25}{81}\times 3}=\frac{5}{9}\sqrt{3}.$$

10. $$2x-5:3x+2=x-1:7x+1$$

By Art. 293, $$(2x-5)(7x+1)=(3x+2)(x-1)$$

$$14x^2-33x-5=3x^2-x-2$$

$$11x^2-32x=3$$

$$121x^2-352x+256=33+256=289$$

$$11x-16=\pm 17$$

$$11x=33 \text{ or } -1$$

$$x=3 \text{ or } -\frac{1}{11}$$

11. $$x^2-4:x^2-9=x^2-5x+6:x^2+4x+3$$

By Art. 293, $(x^2-4)(x^2+4x+3)=(x^2-9)(x^2-5x+6)$

Factoring,

$$(x+2)(x-2)(x+1)(x+3)=(x+3)(x-3)(x-2)(x-3)$$

$$(x-2)(x+3)[(x+2)(x+1)-(x-3)^2]=0$$

$$(x-2)(x+3)(x^2+3x+2-x^2+6x-9)=0$$

$$(x-2)(x+3)(9x-7)=0$$

$$\therefore\ x-2=0, \text{ or } x=2$$

$$x+3=0, \text{ or } x=-3$$

$$9x-7=0, \text{ or } x=\frac{7}{9}$$

12. $$x+\sqrt{1-x^2}:x-\sqrt{1-x^2}=a+\sqrt{b^2-a^2}:a-\sqrt{b^2-a^2}$$

By composition and division,

$$2x:2\sqrt{1-x^2}=2a:2\sqrt{b^2-a^2}$$

Or, $$x:\sqrt{1-x^2}=a:\sqrt{b^2-a^2}$$

Hence, by Art. 293, $$x\sqrt{b^2-a^2}=a\sqrt{1-x^2}$$

$$b^2x^2-a^2x^2=a^2-a^2x^2$$

$$x^2=\frac{a^2}{b^2}$$

$$x=\pm\frac{a}{b}$$

13. $$\begin{cases} x:y=3:5 & (1)\\ x:4=15:y & (2)\end{cases}$$

From (1) and (2) by Art. 293, $3y=5x \quad (3)$

$$xy=60 \quad (4)$$

Dividing (4) by (3), $\frac{x}{3} = \frac{12}{x}$

$$x^2 = 36$$

$$x = \pm 6$$

Substituting in (3), $3y = \pm 30$

$$y = \pm 10$$

14. $$\begin{cases} x + y : x - y = a + b : a - b & (1) \\ x^2 + y^2 = a^2b^2(a^2 + b^2) & (2) \end{cases}$$

From (1), by composition and division,

$$2x : 2y = 2a : 2b$$

Or, $x : y = a : b$

$$ay = bx$$

$$y = \frac{bx}{a} \quad (3)$$

Substituting in (2), $x^2 + \frac{b^2x^2}{a^2} = a^2b^2(a^2 + b^2)$

$$\frac{x^2(a^2 + b^2)}{a^2} = a^2b^2(a^2 + b^2)$$

$$x^2 = a^4b^2$$

$$x = \pm a^2b$$

Substituting in (3), $y = \frac{b}{a}(\pm a^2b) = \pm ab^2$

15. Let x and y represent the numbers; then by the conditions

$$\begin{cases} x : y = \frac{5}{2} : 2 & (1) \\ x - 5 : y - 5 = \frac{4}{3} : 1 & (2) \end{cases}$$

From (1) and (2), by Art. 293, $\frac{5y}{2} = 2x$

or, $y = \frac{4x}{5}$ (3)

and $\frac{4y - 20}{3} = x - 5$

or, $4y - 20 = 3x - 15$ (4)

Substituting from (3) in (4),

$$\frac{16x}{5} - 20 = 3x - 15$$

$$\frac{x}{5} = 5$$

$$x = 25$$

Substituting in (3), $y = 20$

Therefore the numbers are 25 and 20.

16. Let $x =$ the greater part.
Then, $50 - x =$ the less.
By the conditions, $x + 3 : 50 - x - 3 = 3 : 2$

$$2x + 6 = 150 - 3x - 9$$
$$5x = 135$$

Whence, $x = 27$, the greater part,
and $50 - x = 23$, the less.

17. Let $x =$ one part.
Then, $12 - x =$ the other.
By the conditions,

$$x(12 - x) : x^2 + (12 - x)^2 = 3 : 10$$
$$3x^2 + 3(12 - x)^2 = 10x(12 - x)$$
$$3x^2 + 432 - 72x + 3x^2 = 120x - 10x^2$$
$$16x^2 - 192x = -432$$
$$x^2 - 12x = -27$$
$$x^2 - 12x + 36 = 9$$
$$x - 6 = \pm 3$$

Whence, $x = 9$ or 3,
and $12 - x = 3$ or 9
Therefore the parts are 9 and 3.

18. Let x and y represent the numbers; then by the conditions,

$$\begin{cases} x : y = 4 : 9 & (1) \\ x : 12 = 12 : y & (2) \end{cases}$$

From (2), $xy = 144$ (3)
From (1), $4y = 9x$
or, $y = \dfrac{9x}{4}$ (4)
Substituting in (3), $\dfrac{9x^2}{4} = 144$

$$x^2 = 64$$

Whence, $x = 8$
Substituting in (4), $y = 18$
Therefore the numbers are 8 and 18.

19. Let x and y represent the numbers; then by the conditions,

$$\begin{cases} x + y : x - y = 10 : 3 & (1) \\ xy = 364 & (2) \end{cases}$$

From (1), $3x + 3y = 10x - 10y$

$$13y = 7x$$
$$y = \frac{7x}{13} \quad (3)$$

Substituting in (2), $\frac{7x^2}{13} = 364$

$x^2 = 676$

Whence, $x = 26$

Substituting in (3), $y = 14$

Therefore the numbers are 26 and 14.

20. $a - b : b - c = b : c$

By Art. 293, $b^2 - bc = ac - bc$

Or, $b^2 = ac$

Hence b is a mean proportional between a and c.

21. $5a + 4b : 9a + 2b = 5b + 4c : 9b + 2c$

By Art. 293, $(5a + 4b)(9b + 2c) = (9a + 2b)(5b + 4c)$

$$45ab + 10ac + 36b^2 + 8bc = 45ab + 36ac + 10b^2 + 8bc$$

$$26b^2 = 26ac$$

Or, $b^2 = ac$

Hence b is a mean proportional between a and c.

22. $(a + b + c + d)(a - b - c + d) = (a - b + c - d)(a + b - c - d)$

Hence, by Art. 296,

$$a + b + c + d : a + b - c - d = a - b + c - d : a - b - c + d$$

By composition and division,

$$2a + 2b : 2c + 2d = 2a - 2b : 2c - 2d$$

Or, $a + b : c + d = a - b : c - d$

By alternation, $a + b : a - b = c + d : c - d$

By composition and division, $2a : 2b = 2c : 2d$

Or, $a : b = c : d$

23. $ax - by : cx - dy = ay - bz : cy - dz$

By Art. 293, $(ax - by)(cy - dz) = (cx - dy)(ay - bz)$

$$acxy - adxz - bcy^2 + bdyz = acxy - bcxz - ady^2 + bdyz$$

$$(ad - bc)y^2 = (ad - bc)xz$$

$$y^2 = xz$$

Therefore y is a mean proportional between x and z.

24. Let x and y represent the numbers; then by the conditions,

$$\begin{cases} x + 3 : y + 3 = 4 : 3 & (1) \\ x - 8 : y - 8 = 9 : 4 & (2) \end{cases}$$

From (1), $4y + 12 = 3x + 9$

$$y = \frac{3x - 3}{4} \qquad (3)$$

From (2), $9y - 72 = 4x - 32$

$$y = \frac{4x + 40}{9}$$

Equating the values of y, $\frac{3x-3}{4}=\frac{4x+40}{9}$

$$27x-27=16x+160$$

Whence, $x=17$

Substituting in (3), $y=12$

Therefore the numbers are 17 and 12.

25. Let $x=$ the greater number,

and $y=$ the less.

By the conditions,

$$\begin{cases} xy=96 & (1) \\ x^3-y^3:(x-y)^3=19:1 & (2) \end{cases}$$

From (2), $19(x-y)^3=x^3-y^3$

Dividing by $x-y$, $19(x-y)^2=x^2+xy+y^2$

$$19x^2-38xy+19y^2=x^2+xy+y^2$$

$$18x^2+18y^2=39xy$$

$$6x^2+6y^2=13xy=13\times 96, \text{ by (1)}$$

Dividing by 6, $x^2+y^2=13\times 16=208$ (3)

From (1), $2xy=192$ (4)

Adding (3) and (4), $x^2+2xy+y^2=400$

$$x+y=20 \qquad (5)$$

Subtracting (4) from (3),

$$x^2-2xy+y^2=16$$

$$x-y=4 \qquad (6)$$

Adding (5) and (6), $2x=24$

$$x=12$$

Subtracting (6) from (5), $2y=16$

$$y=8$$

Therefore the numbers are 12 and 8.

26. Let $x=$ A's share.

Then, $\frac{9x}{5}=$ B's share,

and $\frac{10}{7}\times\frac{9x}{5}$, or $\frac{18x}{7}=$ C's share.

By the conditions, $x+\frac{9x}{5}+\frac{18x}{7}=564$

$$35x+63x+90x=35\times 564$$

$$188x=35\times 564$$

Whence, $x=35\times 3=105$, A's share.

Therefore, $\frac{9x}{5}=189$, B's share,

and $\frac{18x}{7}=270$, C's share.

27. Let x and y denote the rates of the trains; then by the conditions

$$x+y:x-y=30:2$$

By composition and division,

$$2x:2y=32:28$$

Dividing the terms of the first ratio by 2, and of the second by 4,

$$x:y=8:7$$

28. Let $x:1-x=$ the ratio of wine to water in the first vessel,
and $y:1-y=$ the ratio of wine to water in the second.
Then n measures from the first vessel contain nx measures of wine, and
n measures from the second contain ny measures of wine.
The first mixture contains one measure of wine.

Hence, $$x+y=1 \quad (1)$$

The second mixture contains $\frac{2}{5}$ of 5, or 2 measures of wine.

Hence, $$4x+y=2 \quad (2)$$

Subtracting (1) from (2), $$3x=1$$

$$x=\frac{1}{3}$$

Substituting in (1), $$\frac{1}{3}+y=1$$

$$y=\frac{2}{3}$$

Therefore the ratio of wine to water in the first vessel is $\frac{1}{3}:\frac{2}{3}$, or $1:2$; and in the second $\frac{2}{3}:\frac{1}{3}$, or $2:1$.

29. Let x and $a-x$ represent the parts; then by the conditions,

$$x+b:a-x-b=a+3b:a-3b$$

By composition and division,

$$a:2x+2b-a=2a:6b$$

Dividing the terms of the second ratio by 2,

$$a:2x+2b-a=a:3b$$

Therefore, $$2x+2b-a=3b$$

$$2x=a+b$$

Whence, $$x=\frac{a+b}{2}$$

and $$a-x=a-\frac{a+b}{2}=\frac{a-b}{2}$$

Therefore the parts are $\frac{a+b}{2}$ and $\frac{a-b}{2}$.

CHAPTER XXVII.

Art. 314. — Page 265.

2.

$$a = 1, n = 15, d = 5$$
$$\therefore l = 1 + 14 \times 5 = 1 + 70 = 71$$
$$S = \frac{15}{2}(1 + 71) = 15 \times 36 = 540$$

3.

$$a = 7, n = 20, d = 3 - 7 = -4$$
$$\therefore l = 7 + 19(-4) = 7 - 76 = -69$$
$$S = 10(7 - 69) = 10(-62) = -620$$

4.

$$a = -9, n = 23, d = -6 + 9 = 3$$
$$\therefore l = -9 + 22 \times 3 = -9 + 66 = 57$$
$$S = \frac{23}{2}(-9 + 57) = 23 \times 24 = 552$$

5.

$$a = -5, n = 29, d = -10 + 5 = -5$$
$$\therefore l = -5 + 28(-5) = -5 - 140 = -145$$
$$S = \frac{29}{2}(-5 - 145) = 29(-75) = -2175$$

6.

$$a = \frac{1}{4}, n = 35, d = \frac{1}{4}$$
$$\therefore l = \frac{1}{4} + 34\left(\frac{1}{4}\right) = \frac{35}{4}$$
$$S = \frac{35}{2}\left(\frac{1}{4} + \frac{35}{4}\right) = \frac{35}{2} \times 9 = \frac{315}{2}$$

7.

$$a = \frac{3}{5}, n = 19, d = \frac{8}{15} - \frac{3}{5} = -\frac{1}{15}$$
$$\therefore l = \frac{3}{5} + 18\left(-\frac{1}{15}\right) = \frac{3}{5} - \frac{6}{5} = -\frac{3}{5}$$
$$S = \frac{19}{2}\left(\frac{3}{5} - \frac{3}{5}\right) = 0$$

8.
$$a=\frac{2}{3},\ n=16,\ d=\frac{3}{4}-\frac{2}{3}=\frac{1}{12}$$
$$\therefore\ l=\frac{2}{3}+15\left(\frac{1}{12}\right)=\frac{2}{3}+\frac{15}{12}=\frac{23}{12}$$
$$S=8\left(\frac{2}{3}+\frac{23}{12}\right)=8\left(\frac{31}{12}\right)=\frac{62}{3}$$

9.
$$a=\frac{1}{2},\ n=22,\ d=\frac{5}{11}-\frac{1}{2}=-\frac{1}{22}$$
$$\therefore\ l=\frac{1}{2}+21\left(-\frac{1}{22}\right)=\frac{1}{2}-\frac{21}{22}=-\frac{5}{11}$$
$$S=11\left(\frac{1}{2}-\frac{5}{11}\right)=11\left(\frac{1}{22}\right)=\frac{1}{2}$$

10.
$$a=-3,\ n=17,\ d=-\frac{5}{2}+3=\frac{1}{2}$$
$$\therefore\ l=-3+16\left(\frac{1}{2}\right)=-3+8=5$$
$$S=\frac{17}{2}(-3+5)=\frac{17}{2}\times 2=17$$

11.
$$a=-\frac{2}{5},\ n=14,\ d=\frac{1}{3}+\frac{2}{5}=\frac{11}{15}$$
$$\therefore\ l=-\frac{2}{5}+13\left(\frac{11}{15}\right)=-\frac{2}{5}+\frac{143}{15}=\frac{137}{15}$$
$$S=7\left(-\frac{2}{5}+\frac{137}{15}\right)=7\left(\frac{131}{15}\right)=\frac{917}{15}$$

Art. 315.— Pages 267-269.

4.
$$d=4,\ l=75,\ n=19$$
By (I.),
$$75=a+18\times 4=a+72$$
$$\therefore\ a=3$$
By (II.),
$$S=\frac{19}{2}(3+75)=19\times 39=741$$

5.
$$d=-1,\ n=15,\ S=-\frac{165}{2}$$
By (I.),
$$l=a+14(-1) \qquad (1)$$
By (II.),
$$-\frac{165}{2}=\frac{15}{2}(a+l) \qquad (2)$$
From (1)
$$a-l=14 \qquad (3)$$

From (2), $a + l = -11$ (4)

Adding (3) and (4), $2a = 3$

$$a = \frac{3}{2} = 1\tfrac{1}{2}$$

Subtracting (3) from (4), $2l = -25$

$$l = -\frac{25}{2} = -12\tfrac{1}{2}$$

6. $a = -\frac{2}{3},\ n = 18,\ l = 5$

By (I.), $5 = -\frac{2}{3} + 17d$

$$17d = \frac{17}{3}$$

$$d = \frac{1}{3}$$

By (II.), $S = 9\left(-\frac{2}{3} + 5\right) = -6 + 45 = 39$

7. $a = -\frac{3}{4},\ n = 7,\ S = -7$

By (II.), $-7 = \frac{7}{2}\left(-\frac{3}{4} + l\right)$

$$l - \frac{3}{4} = -2$$

$$\therefore l = -\frac{5}{4} = -1\tfrac{1}{4}$$

By (I.), $-\frac{5}{4} = -\frac{3}{4} + 6d$

$$6d = -\frac{1}{2}$$

$$d = -\frac{1}{12}$$

8. $a = \frac{3}{2},\ l = -\frac{57}{2},\ S = -\frac{351}{2}$

By (II.) $-\frac{351}{2} = \frac{n}{2}\left(\frac{3}{2} - \frac{57}{2}\right)$

$$-351 = -27n$$

$$n = 13$$

By (I.), $-\frac{57}{2} = \frac{3}{2} + 12d$

$$12d = -30$$

$$d = -2\tfrac{1}{2}$$

9. $$l = -31,\ n = 13,\ S = -169$$

By (II.), $$-169 = \frac{13}{2}(a - 31)$$
$$a - 31 = -26$$
$$a = 5$$

By (I.), $$-31 = 5 + 12d$$
$$12d = -36$$
$$d = -3$$

10. $$d = -3,\ S = -328,\ a = 2$$

By (I.), $$l = 2 + (n - 1)(-3) \quad (1)$$

By (II.), $$-328 = \frac{n}{2}(2 + l) \quad (2)$$

From (1), $$l = 5 - 3n \quad (3)$$

Substituting in (2), $$-328 = \frac{n}{2}(7 - 3n)$$
$$3n^2 - 7n = 656$$
$$36n^2 - 84n + 49 = 7872 + 49 = 7921$$
$$6n - 7 = \pm 89$$
$$6n = 96$$
$$n = 16$$

Substituting in (3), $$l = 5 - 48 = -43$$

11. $$a = 3,\ l = \frac{128}{3},\ d = \frac{7}{3}$$

By (I.), $$\frac{128}{3} = 3 + (n - 1)\frac{7}{3}$$
$$\frac{7}{3}(n - 1) = \frac{119}{3}$$
$$n - 1 = 17$$
$$n = 18$$

By (II.), $$S = 9\left(3 + \frac{128}{3}\right) = 27 + 384 = 411$$

12. $$d = -4,\ n = 17,\ S = -493$$

By Art. 313, $$-493 = \frac{17}{2}[2a + 16(-4)]$$
$$2a - 64 = -58$$
$$2a = 6$$
$$a = 3$$

By (I.), $$l = 3 + 16(-4) = -61$$

13. $l=\frac{7}{2},\ d=\frac{1}{3},\ S=20$

By (I.), $\frac{7}{2}=a+(n-1)\frac{1}{3}$ (1)

By (II.), $20=\frac{n}{2}\left(a+\frac{7}{2}\right)$ (2)

From (1), $a=\frac{1-n}{3}+\frac{7}{2}$ (3)

Substituting in (2), $20=\frac{n}{2}\left(7+\frac{1-n}{3}\right)$

$$120=22n-n^2$$
$$n^2-22n+121=1$$
$$n-11=\pm 1$$
$$n=10 \text{ or } 12$$

Substituting in (3), $a=-3+\frac{7}{2} \text{ or } -\frac{11}{3}+\frac{7}{2}$

$$=\frac{1}{2} \text{ or } -\frac{1}{6}$$

14. $l=\frac{79}{2},\ n=21,\ S=\frac{819}{2}$

By (II.), $\frac{819}{2}=\frac{21}{2}\left(a+\frac{79}{2}\right)$

$$a+\frac{79}{2}=39$$
$$a=-\frac{1}{2}$$

By (I.), $\frac{79}{2}=-\frac{1}{2}+20d$

$$20d=40$$
$$d=2$$

15. $a=-\frac{1}{3},\ l=-\frac{4}{3},\ S=-\frac{40}{3}$

By (II.), $-\frac{40}{3}=\frac{n}{2}\left(-\frac{1}{3}-\frac{4}{3}\right)$

$$-\frac{40}{3}=-\frac{5n}{6}$$
$$n=16$$

By (I.), $-\frac{4}{3}=-\frac{1}{3}+15d$

$$15d=-1$$
$$d=-\frac{1}{15}$$

16. $$a=-\frac{3}{4},\ n=15,\ S=120$$

By (II.), $$120=\frac{15}{2}\left(-\frac{3}{4}+l\right)$$

$$l-\frac{3}{4}=16$$

$$l=\frac{67}{4}$$

By (I.), $$\frac{67}{4}=-\frac{3}{4}+14d$$

$$14d=\frac{70}{4}$$

$$d=\frac{5}{4}$$

17. $$l=-47,\ d=-1,\ S=-1118$$

By (I.), $$-47=a+(n-1)(-1) \qquad (1)$$

By (II.), $$-1118=\frac{n}{2}(a-47) \qquad (2)$$

From (1), $$a=n-48 \qquad (3)$$

Substituting in (2), $$\frac{n}{2}(n-95)=-1118$$

$$n^2-95n=-2236$$

$$4n^2-380n+9025=-8944+9025=81$$

$$2n-95=\pm 9$$

$$2n=104 \text{ or } 86$$

$$n=52 \text{ or } 43$$

Substituting in (3), $$a=52-48 \text{ or } 43-48$$

$$=4 \text{ or } -5$$

18. $$a=6,\ d=-\frac{5}{3},\ S=-\frac{203}{3}$$

By (I.), $$l=6+(n-1)\left(-\frac{5}{3}\right) \qquad (1)$$

By (II.), $$-\frac{203}{3}=\frac{n}{2}(6+l) \qquad (2)$$

From (1), $$l=\frac{23-5n}{3} \qquad (3)$$

Substituting in (2), $$-\frac{203}{3}=\frac{n}{2}\left(6+\frac{23-5n}{3}\right)$$

$$-406=41n-5n^2$$

$$100n^2-820n=8120$$

$$100n^2 - 820n + 41^2 = 9801$$
$$10n - 41 = \pm 99$$
$$10n = 140$$
$$n = 14$$

Substituting in (3), $l = \dfrac{23 - 70}{3} = -15\frac{2}{3}$

20. By (I.), $l = a + (n-1)d$

$$(n-1)d = l - a$$
$$\therefore d = \frac{l-a}{n-1}$$

21. By (II.), $S = \dfrac{n}{2}(a + l)$

$$2S = an + nl$$
$$nl = 2S - an$$
$$\therefore l = \frac{2S - an}{n}$$

Substituting in (I.), $\dfrac{2S - an}{n} = a + (n-1)d$

$$2S - an = an + n(n-1)d$$
$$n(n-1)d = 2S - 2an$$
$$\therefore d = \frac{2(S - an)}{n(n-1)}$$

22. By (I.) $l = a + (n-1)d$

$$l - a = (n-1)d$$

Or, $nl - an = n(n-1)d$ (1)

By (II.), $S = \dfrac{n}{2}(a + l)$

Or, $an + nl = 2S$ (2)

Subtracting (1) from (2), $2an = 2S - n(n-1)d$

$$\therefore a = \frac{2S - n(n-1)d}{2n}$$

Adding (1) and (2), $2nl = 2S + n(n-1)d$

$$\therefore l = \frac{2S + n(n-1)d}{2n}$$

23. By (I.), $l = a + (n-1)d$

$$(n-1)d = l - a$$
$$n - 1 = \frac{l-a}{d}$$

$$\therefore\ n=\frac{l-a}{d}+1$$
$$=\frac{l-a+d}{d}$$

Substituting in (II.), $$S=\frac{l-a+d}{2d}(a+l)$$
$$=\frac{(l+a)(l-a+d)}{2d}$$

24. By (I.), $$l=a+(n-1)d$$
$$\therefore\ a=l-(n-1)d$$

Substituting in (II.), $$S=\frac{n}{2}[l-(n-1)d+l]$$
$$=\frac{n}{2}[2l-(n-1)d]$$

25. By (II.), $$S=\frac{n}{2}(a+l)$$
$$2S=an+nl$$
$$an=2S-nl$$
$$\therefore\ a=\frac{2S-nl}{n}$$

Substituting in (I.), $$l=\frac{2S-nl}{n}+(n-1)d$$
$$nl=2S-nl+n(n-1)d$$
$$n(n-1)d=2nl-2S$$
$$\therefore\ d=\frac{2(nl-S)}{n(n-1)}$$

26. From the result in Ex. 19,
$$n=\frac{d-2a\pm\sqrt{8dS+(2a-d)^2}}{2d}$$

Substituting in (I.), $$l=a+\left[\frac{d-2a\pm\sqrt{8dS+(2a-d)^2}}{2d}-1\right]d$$
$$=\frac{2a+d-2a\pm\sqrt{8dS+(2a-d)^2}-2d}{2}$$
$$=\frac{-d\pm\sqrt{8dS+(2a-d)^2}}{2}$$

27. By (II.), $$S=\frac{n}{2}(a+l)$$

$$2S=n(a+l)$$

$$\therefore n=\frac{2S}{a+l}$$

Substituting in (I.), $$l=a+\left(\frac{2S}{a+l}-1\right)d$$

$$l-a=\frac{(2S-a-l)d}{a+l}$$

$$(2S-a-l)d=l^2-a^2$$

$$\therefore d=\frac{l^2-a^2}{2S-a-l}$$

28. By (I.), $$l=a+(n-1)d$$

$$\therefore a=l-(n-1)d \qquad (1)$$

Substituting in (II.), $$S=\frac{n}{2}[2l-(n-1)d]$$

$$2S=2nl-dn^2+dn$$

$$dn^2-(2l+d)n=-2S$$

$$4d^2n^2-4d(2l+d)n+(2l+d)^2=(2l+d)^2-8dS$$

$$2dn-(2l+d)=\pm\sqrt{(2l+d)^2-8dS}$$

$$\therefore n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}$$

Substituting in (1), $$a=l-\left[\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}-1\right]d$$

$$=\frac{2l-2l-d\mp\sqrt{(2l+d)^2-8dS}+2d}{2}$$

$$=\frac{d\mp\sqrt{(2l+d)^2-8dS}}{2}$$

Art. 316.—Page 270.

2. $$a=2,\ l=4,\ n=7$$

Substituting in (I.), $$4=2+6d$$

$$6d=2$$

$$d=\frac{1}{3}$$

Hence the series is

$$2,\ 2\tfrac{1}{3},\ 2\tfrac{2}{3},\ 3,\ 3\tfrac{1}{3},\ 3\tfrac{2}{3},\ 4$$

3. $$a = 3,\ l = -1,\ n = 9$$

Substituting in (I.),
$$-1 = 3 + 8d$$
$$8d = -4$$
$$d = -\frac{1}{2}$$

Hence the series is
$$3,\ 2\tfrac{1}{2},\ 2,\ 1\tfrac{1}{2},\ 1,\ \tfrac{1}{2},\ 0,\ -\tfrac{1}{2},\ -1.$$

4. $$a = -1,\ l = -7,\ n = 6$$

Substituting in (I.),
$$-7 = -1 + 5d$$
$$5d = -6$$
$$d = -1\tfrac{1}{5}$$

Hence the series is
$$-1,\ -2\tfrac{1}{5},\ -3\tfrac{2}{5},\ -4\tfrac{3}{5},\ -5\tfrac{4}{5},\ -7.$$

5. $$a = -8,\ l = -4,\ n = 8$$

Substituting in (I.),
$$-4 = -8 + 7d$$
$$7d = 4$$
$$d = \frac{4}{7}$$

Hence the series is
$$-8,\ -7\tfrac{3}{7},\ -6\tfrac{6}{7},\ -6\tfrac{2}{7},\ -5\tfrac{5}{7},\ -5\tfrac{1}{7},\ -4\tfrac{4}{7},\ -4.$$

6. $$a = \frac{1}{2},\ l = -\frac{13}{10},\ n = 10$$

Substituting in (I.),
$$-\frac{13}{10} = \frac{1}{2} + 9d$$
$$9d = -\frac{18}{10}$$
$$d = -\frac{1}{5}$$

Hence the series is
$$\frac{1}{2},\ \frac{3}{10},\ \frac{1}{10},\ -\frac{1}{10},\ -\frac{3}{10},\ -\frac{1}{2},\ -\frac{7}{10},\ -\frac{9}{10},\ -\frac{11}{10},\ -\frac{13}{10}.$$

7. Result,
$$\frac{\frac{7}{3} - \frac{9}{5}}{2} = \frac{4}{15}.$$

8. Result,
$$\frac{(a+b)^2 - (a-b)^2}{2} = \frac{a^2 + 2ab + b^2 - a^2 + 2ab - b^2}{2}$$
$$= 2ab.$$

9. Result,
$$\frac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{2}=\frac{(a+b)^2+(a-b)^2}{2(a^2-b^2)}$$
$$=\frac{a^2+2ab+b^2+a^2-2ab+b^2}{2(a^2-b^2)}$$
$$=\frac{2a^2+2b^2}{2(a^2-b^2)}=\frac{a^2+b^2}{a^2-b^2}.$$

Art. 317. — Pages 271, 272.

3. $$a=1,\ l=99,\ n=50$$
By (II.), $$S=25(1+99)=2500$$

4. $$\begin{cases} a+6d=27 & (1)\\ a+12d=-3 & (2)\end{cases}$$
Subtracting (1) from (2), $$6d=-30$$
$$d=-5$$
$$\therefore 8d=-40 \qquad (3)$$
Adding (2) and (3), $$a+20d=-43$$
Hence the twenty-first term is -43.

5. Let the numbers be $x-3y$, $x-y$, $x+y$, and $x+3y$.

By the conditions, $$\begin{cases} 2x-2y=22, \text{ or } x-y=11 & (1)\\ 2x+2y=36, \text{ or } x+y=18 & (2)\end{cases}$$
Subtracting (1) from (2), $$2y=7 \qquad (3)$$
Subtracting (3) from (1), $$x-3y=4 \qquad (4)$$
Adding (2) and (3), $$x+3y=25 \qquad (5)$$
Hence by (4), (1), (2), and (5), the numbers are 4, 11, 18, and 25.

6. Let $$x=\text{the number of years.}$$
$$a=270,\ d=-25,\ n=x$$
By Art. 313, $$S=\frac{x}{2}[540-25(x-1)]$$
Then by the conditions, $$145x=\frac{x}{2}[540-25(x-1)]$$
$$290=540-25x+25$$
$$25x=275$$
$$x=11$$

7. $$a=m,\ d=m-3n,\ n=10$$
By (I.), $$l=m+9(m-3n)$$
$$=10m-27n$$
By (II.), $$S=5(m+10m-27n)$$
$$=55m-135n$$

8. Let a' be the first term, and d the common difference; then

$$\begin{cases} a' + 6d = 5a + 4b & (1) \\ a' + 18d = 9a - 2b & (2) \end{cases}$$

Subtracting (1) from (2), $12d = 4a - 6b$

$$\therefore 4d = \frac{4a}{3} - 2b \qquad (3)$$

Subtracting (3) from (2), $a' + 14d = \frac{23a}{3}$

Hence the fifteenth term is $\frac{23a}{3}$.

9. $a = 2,\ l = 500,\ n = 250$

By (II.), $S = 125(2 + 500) = 62750$

10. Let the numbers be $x - 3y$, $x - y$, $x + y$, and $x + 3y$.
Then by the conditions,

$$\begin{cases} (x - 3y)^2 + (x + 3y)^2 = 200 & (1) \\ (x - y)^2 + (x + y)^2 = 136 & (2) \end{cases}$$

From (1), $2x^2 + 18y^2 = 200$ (3)

From (2), $2x^2 + 2y^2 = 136$ (4)

Subtracting (4) from (3), $16y^2 = 64$

$$y = \pm 2$$

Substituting in (4), $2x^2 + 8 = 136$

$$2x^2 = 128$$

$$x = \pm 8$$

Therefore the numbers are

$$\pm 8 \mp 6,\ \pm 8 \mp 2,\ \pm 8 \pm 2, \text{ and } \pm 8 \pm 6.$$

That is, 2, 6, 10, and 14; or, -2, -6, -10, and -14.

11.
$$\begin{cases} a + 6d = -\frac{1}{2} & (1) \\ a + 12d = \frac{3}{2} & (2) \\ a + (n-1)d = \frac{9}{2} & (3) \end{cases}$$

Subtracting (1) from (2), $6d = 2$

$$d = \frac{1}{3} \qquad (4)$$

Subtracting (2) from (3),

$$(n - 13)d = 3 \qquad (5)$$

Substituting from (4) in (5),

$$\frac{n - 13}{3} = 3$$

$$n = 22$$

12. Let the quantities be $x-2y$, $x-y$, x, $x+y$, and $x+2y$.
Then by the conditions,

$$\begin{cases} x-2y+x+x+y=3 & (1) \\ (x-y)(x+2y)=-8 & (2) \end{cases}$$

From (2), $$x^2+xy-2y^2=-8 \quad (3)$$

From (1), $$y=3x-3 \quad (4)$$

Substituting in (3),

$$x^2+x(3x-3)-2(3x-3)^2=-8$$
$$x^2+3x^2-3x-18x^2+36x-18=-8$$
$$14x^2-33x=-10$$
$$784x^2-56\times 33x=-560$$
$$784x^2-56\times 33x+1089=529$$
$$28x-33=\pm 23$$
$$28x=56 \text{ or } 10$$
$$x=2 \text{ or } \frac{5}{14}$$

Substituting in (4), $$y=3 \text{ or } -\frac{27}{14}$$

Hence the numbers are

$$2-6,\ 2-3,\ 2,\ 2+3, \text{ and } 2+6;$$

or, $$\frac{5}{14}+\frac{54}{14},\ \frac{5}{14}+\frac{27}{14},\ \frac{5}{14},\ \frac{5}{14}-\frac{27}{14}, \text{ and } \frac{5}{14}-\frac{54}{14}.$$

That is, $-4,\ -1,\ 2,\ 5,$ and 8;

or, $$\frac{59}{14},\ \frac{16}{7},\ \frac{5}{14},\ -\frac{11}{7}, \text{ and } -\frac{7}{2}.$$

13. Let $x=$ the number of days.

$$a=8,\ d=\frac{1}{2},\ n=x$$

By Art. 313, $$S=\frac{x}{2}\left[16+(x-1)\frac{1}{2}\right]$$

Then by the conditions,

$$\frac{x}{2}\left[16+\frac{x-1}{2}\right]=10x$$
$$16+\frac{x-1}{2}=20$$
$$\frac{x-1}{2}=4$$

Whence, $$x=9$$
and $$10x=90$$

Therefore they will meet after 9 days, at a distance of 90 leagues.

14. $$a = 16\tfrac{1}{12},\ d = 32\tfrac{1}{6},\ n = 16$$

By Art. 313,
$$\begin{aligned} S &= 8[32\tfrac{1}{6} + 15\,(32\tfrac{1}{6})] \\ &= 128 \times 32\tfrac{1}{6} \\ &= \frac{12352}{3} = 4117\tfrac{1}{3} \end{aligned}$$

15. Let the quantities be $x - y$, x, and $x + y$.

Then by the conditions,
$$\left\{\begin{aligned} (x-y)^2 + (x+y)^2 - x &= 123 \qquad (1) \\ x - \frac{x-y}{3} &= 6 \qquad (2) \end{aligned}\right.$$

From (1), $$2x^2 + 2y^2 - x = 123 \qquad (3)$$

From (2), $$2x + y = 18$$
$$y = 18 - 2x \qquad (4)$$

Substituting in (3),
$$\begin{aligned} 2x^2 + 648 - 144x + 8x^2 - x &= 123 \\ 10x^2 - 145x &= -525 \\ 2x^2 - 29x &= -105 \\ 16x^2 - 232x + 841 &= -840 + 841 = 1 \\ 4x - 29 &= \pm 1 \\ 4x &= 28 \text{ or } 30 \\ x &= 7 \text{ or } \frac{15}{2} \end{aligned}$$

Substituting in (4), $$y = 4 \text{ or } 3$$

Hence the numbers are
$$3,\ 7, \text{ and } 11;\ \text{ or, } \frac{9}{2},\ \frac{15}{2}, \text{ and } \frac{21}{2}.$$

16. Let $x =$ the number of hours.
$$a = 4\tfrac{1}{2},\ d = \frac{1}{4},\ n = x$$

By Art. 313, $$S = \frac{x}{2}\left[9 + \frac{x-1}{4}\right]$$

Then by the conditions,
$$\begin{aligned} \frac{x}{2}\left[9 + \frac{x-1}{4}\right] &= 4x + 11 \\ \frac{9x}{2} + \frac{x(x-1)}{8} &= 4x + 11 \\ 36x + x^2 - x &= 32x + 88 \\ x^2 + 3x &= 88 \\ 4x^2 + 12x + 9 &= 352 + 9 = 361 \\ 2x + 3 &= \pm 19 \\ 2x &= 16 \end{aligned}$$

Whence, $$x = 8$$

17. The interest for the second year is \$5; for the third year \$10; etc.

Hence, $a = 5,\ d = 5,\ n = 19$

By Art. 313, $S = \frac{19}{2}[10 + 18 \times 5] = 950,$

which is the total amount received for interest.

The total amount of savings is $20 \times \$100$, or \$2000.

Therefore the entire property will amount to \$2000 + \$950, or \$2950.

18. Let $x - y$, x, and $x + y$ represent the first, second, and third digits.

Then, $100(x - y) + 10x + x + y$, or $111x - 99y$

will represent the number; and

$100(x + y) + 10x + x - y$, or $111x + 99y$

will represent the number with its digits inverted.

By the conditions,

$$\begin{cases} x - y = x + x + y + 1 & (1) \\ 111x - 99y - 594 = 111x + 99y & (2) \end{cases}$$

From (2), $198y = -594$

$y = -3$

Substituting in (1), $x + 3 = 2x - 3 + 1$

$x = 5$

Therefore, $111x - 99y = 555 + 297$

$= 852$, the number required.

CHAPTER XXVIII.

Art. 321.—Pages 274, 275.

3.
$$a=1,\ n=9,\ r=2$$
$$\therefore\ l=2^8=256$$
$$S=\frac{2\times256-1}{2-1}=511$$

4.
$$a=3,\ n=7,\ r=\frac{2}{3}$$
$$\therefore\ l=3\left(\frac{2}{3}\right)^6=\frac{2^6}{3^5}=\frac{64}{243}$$
$$S=\frac{\frac{2}{3}\times\frac{64}{243}-3}{\frac{2}{3}-1}=\frac{-\frac{2059}{729}}{-\frac{1}{3}}=\frac{2059}{243}$$

5.
$$a=-2,\ n=6,\ r=\frac{8}{-2}=-4.$$
$$\therefore\ l=-2(-4)^5=-2(-1024)=2048$$
$$S=\frac{-4\times2048+2}{-4-1}=\frac{-8190}{-5}=1638$$

6.
$$a=2,\ n=10,\ r=-\frac{1}{2}$$
$$\therefore\ l=2\left(-\frac{1}{2}\right)^9=-\frac{1}{2^8}=-\frac{1}{256}$$
$$S=\frac{-\frac{1}{2}\left(-\frac{1}{256}\right)-2}{-\frac{1}{2}-1}=\frac{-\frac{1023}{512}}{-\frac{3}{2}}=\frac{341}{256}$$

7.
$$a=\frac{1}{2},\ n=11,\ r=\frac{1}{2}$$
$$\therefore\ l=\frac{1}{2}\left(\frac{1}{2}\right)^{10}=\frac{1}{2^{11}}=\frac{1}{2048}$$
$$S=\frac{\frac{1}{2}\times\frac{1}{2048}-\frac{1}{2}}{\frac{1}{2}-1}=1-\frac{1}{2048}=\frac{2047}{2048}$$

8.
$$a=\frac{2}{3},\ n=8,\ r=\frac{-1}{\frac{2}{3}}=-\frac{3}{2}$$
$$\therefore\ l=\frac{2}{3}\left(-\frac{3}{2}\right)^7=-\frac{3^6}{2^6}=-\frac{729}{64}$$
$$S=\frac{-\frac{3}{2}\left(-\frac{729}{64}\right)-\frac{2}{3}}{-\frac{3}{2}-1}=\frac{\frac{6305}{384}}{-\frac{5}{2}}=-\frac{1261}{192}$$

9.
$$a=8,\ n=9,\ r=\frac{1}{2}$$
$$\therefore\ l=8\left(\frac{1}{2}\right)^8=\frac{1}{2^5}=\frac{1}{32}$$
$$S=\frac{\frac{1}{2}\times\frac{1}{32}-8}{\frac{1}{2}-1}=\frac{-\frac{511}{64}}{-\frac{1}{2}}=\frac{511}{32}$$

10.
$$a=\frac{3}{4},\ n=6,\ r=-\frac{1}{3}$$
$$\therefore\ l=\frac{3}{4}\left(-\frac{1}{3}\right)^5=-\frac{1}{4\times 3^4}=-\frac{1}{324}$$
$$S=\frac{-\frac{1}{3}\left(-\frac{1}{324}\right)-\frac{3}{4}}{-\frac{1}{3}-1}=\frac{-\frac{728}{972}}{-\frac{4}{3}}=\frac{91}{162}$$

11.
$$a=3,\ n=7,\ r=\frac{-6}{3}=-2$$
$$\therefore\ l=3\,(-2)^6=3\times 64=192$$
$$S=\frac{-2\times 192-3}{-2-1}=\frac{-387}{-3}=129$$

12.
$$a=-\frac{2}{3},\ n=10,\ r=\frac{1}{2}$$
$$\therefore\ l=-\frac{2}{3}\left(\frac{1}{2}\right)^9=-\frac{1}{3\times 2^8}=-\frac{1}{768}$$
$$S=\frac{\frac{1}{2}\left(-\frac{1}{768}\right)+\frac{2}{3}}{\frac{1}{2}-1}=\frac{\frac{1023}{1536}}{-\frac{1}{2}}=-\frac{341}{256}$$

Art. 322. — Pages 276, 277.

3. $$r=2,\ n=10,\ l=256$$

By (I.), $$256=a\,(2)^9=512a$$

$$\therefore\ a=\frac{1}{2}$$

By (II.), $$S=\frac{2\times 256-\frac{1}{2}}{2-1}=\frac{1023}{2}$$

4. $$r=-2,\ n=6,\ S=\frac{63}{2}$$

By (I.), $$l=a\,(-2)^5 \qquad (1)$$

By (II.), $$\frac{63}{2}=\frac{-2\,l-a}{-2-1} \qquad (2)$$

From (2), $$4\,l+2\,a=189 \qquad (3)$$

From (1), $$l=-32\,a \qquad (4)$$

Substituting in (3), $$-128\,a+2\,a=189$$

$$-126\,a=189$$

$$\therefore\ a=-\frac{3}{2}$$

Substituting in (4), $$l=48$$

5. $$a=2,\ n=7,\ l=1458$$

By (I.), $$1458=2r^6$$

$$r^6=729$$

$$\therefore\ r=\pm 3$$

By (II.), $$S=\frac{\pm 3\times 1458-2}{\pm 3-1}$$

$$=\frac{4374-2}{2}\text{ or }\frac{-4374-2}{-4}$$

$$=2186\text{ or }1094$$

6. $$a=1,\ r=3,\ l=81$$

By (I.), $$81=3^{n-1}$$

$$n-1=4$$

$$\therefore\ n=5$$

By (II.), $$S=\frac{3\times 81-1}{3-1}$$

$$=\frac{243-1}{2}=121$$

7.
$$r=\frac{1}{3},\ n=8,\ S=\frac{6560}{6561}$$

By (I.),
$$l=a\left(\frac{1}{3}\right)^7 \qquad (1)$$

By (II.),
$$\frac{6560}{6561}=\frac{\frac{1}{3}l-a}{\frac{1}{3}-1} \qquad (2)$$

From (2),
$$l-3a=-\frac{13120}{6561} \qquad (3)$$

From (1),
$$a=2187\,l \qquad (4)$$

Substituting in (3),
$$l-6561\,l=-\frac{13120}{6561}$$
$$6560\,l=\frac{13120}{6561}$$
$$\therefore\ l=\frac{2}{6561}$$

Substituting in (4),
$$a=2187\left(\frac{2}{6561}\right)=\frac{2}{3}$$

8. $a=3,\ n=6,\ l=-\frac{3}{1024}$

By (I.), $-\frac{3}{1024}=3r^5$
$$r^5=-\frac{1}{1024}$$
$$\therefore\ r=-\frac{1}{4}$$

By (II.),
$$S=\frac{-\frac{1}{4}\left(-\frac{3}{1024}\right)-3}{-\frac{1}{4}-1}$$
$$=\frac{-\frac{12285}{4096}}{-\frac{5}{4}}=\frac{2457}{1024}$$

9. $a=2,\ l=\frac{1}{32},\ S=\frac{127}{32}$

By (II.),
$$\frac{127}{32}=\frac{\frac{r}{32}-2}{r-1}$$
$$127r-127=r-64$$
$$126r=63$$
$$\therefore\ r=\frac{1}{2}$$

By (I.),
$$\frac{1}{32}=2\left(\frac{1}{2}\right)^{n-1}$$
$$\left(\frac{1}{2}\right)^{n-1}=\frac{1}{64}$$
$$n-1=6$$
$$\therefore\ n=7$$

10. $a=\frac{1}{2},\ r=-3,\ S=-91$

By (II.),
$$-91=\frac{-3l-\frac{1}{2}}{-3-1}$$
$$3l+\frac{1}{2}=-364$$
$$6l=-729$$
$$\therefore\ l=-\frac{243}{2}$$

By (I.), $-\frac{243}{2}=\frac{1}{2}(-3)^{n-1}$

$$(-3)^{n-1}=-243$$
$$n-1=5$$
$$\therefore n=6$$

11. $l=-128,\ r=2,\ S=-255$

By (II.), $-255=\frac{2(-128)-a}{2-1}$

$$-255=-256-a$$
$$\therefore a=-1$$

By (I.), $-128=-(2)^{n-1}$

$$2^{n-1}=128$$
$$n-1=7$$
$$\therefore n=8$$

12. By (II.), $S=\frac{rl-a}{r-1}$

$$(r-1)S=rl-a$$
$$\therefore l=\frac{a+(r-1)S}{r}$$

13. By (II.), $S=\frac{rl-a}{r-1}$

$$rS-S=rl-a$$
$$rS-rl=S-a$$
$$\therefore r=\frac{S-a}{S-l}$$

14. By (II.), $S=\frac{rl-a}{r-1}$

$$(r-1)S=rl-a$$
$$\therefore a=rl-(r-1)S$$

15. By (I.), $l=ar^{n-1}$

$$\therefore a=\frac{l}{r^{n-1}}$$

Substituting in (II.),

$$S=\frac{rl-\frac{l}{r^{n-1}}}{r-1}$$
$$=\frac{l(r^n-1)}{r^{n-1}(r-1)}$$

16. By (I.), $l=ar^{n-1}$ (1)

Substituting in (II.),

$$S=\frac{ar^n-a}{r\ \ 1}$$
$$(r-1)S=a(r^n-1)$$
$$\therefore a=\frac{(r-1)S}{r^n-1}$$

Substituting in (1),

$$l=\frac{r^{n-1}(r-1)S}{r^n-1}$$

17. By (I.), $l=ar^{n-1}$

$$r^{n-1}=\frac{l}{a}$$
$$\therefore r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}$$

Substituting in (II.),

$$S=\frac{l\left(\frac{l}{a}\right)^{\frac{1}{n-1}}-a}{\left(\frac{l}{a}\right)^{\frac{1}{n-1}}-1}$$
$$=\frac{l^{\frac{n}{n-1}}-a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}}-a^{\frac{1}{n-1}}}$$

Art. 323.—Page 278.

2. $a=2,\ r=\frac{1}{2}$

$$\therefore S=\frac{2}{1-\frac{1}{2}}=4$$

3. $a=4,\ r=\frac{-2}{4}=-\frac{1}{2}$

$$\therefore S=\frac{4}{1+\frac{1}{2}}=\frac{8}{3}$$

4. $a=-1,\ r=-\frac{1}{3}$

$\therefore S=\frac{-1}{1+\frac{1}{3}}=-\frac{3}{4}$

5. $a=-3,\ r=\frac{1}{5}$

$\therefore S=\frac{-3}{1-\frac{1}{5}}=-\frac{15}{4}$

6. $a=\frac{3}{4},\ r=\frac{2}{3}$

$\therefore S=\frac{\frac{3}{4}}{1-\frac{2}{3}}=\frac{9}{4}$

7. $a=3,\ r=-\frac{1}{10}$

$\therefore S=\frac{3}{1+\frac{1}{10}}=\frac{30}{11}$

8. $a=-8,\ r=\frac{1}{20}$

$\therefore S=\frac{-8}{1-\frac{1}{20}}=-\frac{160}{19}$

9. $a=1,\ r=-\frac{a^2}{x^2}$

$\therefore S=\frac{1}{1+\frac{a^2}{x^2}}=\frac{x^2}{a^2+x^2}$

Art. 324.—Page 279.

2. $.7272\cdots=.72+.0072+\cdots$

$a=.72,\ r=.01$

$\therefore S=\frac{.72}{1-.01}=\frac{.72}{.99}=\frac{8}{11}$

3. $.407407\cdots$

$=.407+.000407+\cdots$

$a=.407,\ r=.001$

$\therefore S=\frac{.407}{1-.001}=\frac{.407}{.999}=\frac{11}{27}$

4. $.7333\cdots=.7+.03+.003+\cdots$

$a=.03,\ r=.1$

$\therefore S=\frac{.03}{1-.1}=\frac{.03}{.9}=\frac{1}{30}$

$.7+\frac{1}{30}=\frac{22}{30}=\frac{11}{15}$

5. $.52121\cdots$

$=.5+.021+.00021+\cdots$

$a=.021,\ r=.01$

$\therefore S=\frac{.021}{1-.01}=\frac{.021}{.99}=\frac{7}{330}$

$.5+\frac{7}{330}=\frac{172}{330}=\frac{86}{165}$

6. $.110303\cdots$

$=.11+.0003+.000003+\cdots$

$a=.0003,\ r=.01$

$\therefore S=\frac{.0003}{1-.01}=\frac{.0003}{.99}=\frac{1}{3300}$

$.11+\frac{1}{3300}=\frac{364}{3300}=\frac{91}{825}$

7. $.215454\cdots$

$=.21+.0054+.000054+\cdots$

$a=.0054,\ r=.01$

$\therefore S=\frac{.0054}{1-.01}=\frac{.0054}{.99}=\frac{6}{1100}$

$.21+\frac{6}{1100}=\frac{237}{1100}$

Art. 325.—Pages 279, 280.

2. $$a = 3,\ l = \frac{128}{729},\ n = 8$$

$\therefore$ by (I.), $$\frac{128}{729} = 3r^7$$

$$r^7 = \frac{128}{2187}$$

$$\therefore\ r = \frac{2}{3}$$

Hence the series is

$$3,\ 2,\ \frac{4}{3},\ \frac{8}{9},\ \frac{16}{27},\ \frac{32}{81},\ \frac{64}{243},\ \frac{128}{729}.$$

3. $$a = \frac{1}{2},\ l = 364\tfrac{1}{2},\ n = 7$$

$\therefore$ by (I.), $$\frac{729}{2} = \frac{1}{2}r^6$$

$$r^6 = 729$$

$$\therefore\ r = \pm 3$$

Hence the series are

$$\frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{27}{2},\ \frac{81}{2},\ \frac{243}{2},\ \frac{729}{2};$$

and $$\frac{1}{2},\ -\frac{3}{2},\ \frac{9}{2},\ -\frac{27}{2},\ \frac{81}{2},\ -\frac{243}{2},\ \frac{729}{2}.$$

4. $$a = -2,\ l = -4374,\ n = 8$$

$\therefore$ by (I.), $$-4374 = -2r^7$$

$$r^7 = 2187$$

$$\therefore\ r = 3$$

Hence the series is

$$-2,\ -6,\ -18,\ -54,\ -162,\ -486,\ -1458,\ -4374.$$

5. $$a = \frac{3}{2},\ l = \frac{3}{512},\ n = 9$$

$\therefore$ by (I.), $$\frac{3}{512} = \frac{3}{2}r^8$$

$$r^8 = \frac{1}{256}$$

$$r = \pm\frac{1}{2}$$

Hence the series are

$$\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \frac{3}{64}, \frac{3}{128}, \frac{3}{256}, \frac{3}{512};$$

and $$\frac{3}{2}, -\frac{3}{4}, \frac{3}{8}, -\frac{3}{16}, \frac{3}{32}, -\frac{3}{64}, \frac{3}{128}, -\frac{3}{256}, \frac{3}{512}.$$

6. $$a = -2,\ l = -128,\ n = 7$$

$\therefore$ by (I.), $$-128 = -2r^6$$

$$r^6 = 64$$

$$\therefore r = \pm 2$$

Hence the series are

$$-2, -4, -8, -16, -32, -64, -128;$$

and $$-2, 4, -8, 16, -32, 64, -128.$$

7. $$a = 3,\ l = -\frac{729}{1024},\ n = 6$$

$\therefore$ by (I.), $$-\frac{729}{1024} = 3r^5$$

$$r^5 = -\frac{243}{1024}$$

$$\therefore r = -\frac{3}{4}$$

Hence the series is

$$3, -\frac{9}{4}, \frac{27}{16}, -\frac{81}{64}, \frac{243}{256}, -\frac{729}{1024}.$$

8. Result, $$\sqrt{\frac{35}{3} \times \frac{15}{7}} = \sqrt{25} = 5.$$

9. Result, $$\sqrt{(4x^2 + 12xy + 9y^2)(4x^2 - 12xy + 9y^2)}$$
$$= (2x + 3y)(2x - 3y) = 4x^2 - 9y^2.$$

10. Result, $$\sqrt{\left(\frac{a^2 - ab}{ab + b^2} \times \frac{a^2 + ab}{ab - b^2}\right)}$$
$$= \sqrt{\left(\frac{a(a-b)}{b(a+b)} \times \frac{a(a+b)}{b(a-b)}\right)}$$
$$= \sqrt{\frac{a^2}{b^2}} = \frac{a}{b}.$$

Art. 326. — Pages 280, 281.

2.
$$\begin{cases} ar^4 = 48 & (1) \\ ar^7 = -384 & (2) \end{cases}$$

Dividing (2) by (1), $r^3 = -8$

$\therefore r = -2$

Substituting in (1), $16a = 48$

$\therefore a = 3$

3. Let the quantities be a, ar, ar^2, and ar^3; then by the conditions,
$$\begin{cases} a + ar = 15 & (1) \\ ar^2 + ar^3 = 60 & (2) \end{cases}$$

Dividing (2) by (1), $r^2 = 4$

Whence, $r = \pm 2$

Substituting in (1), $a \pm 2a = 15$

Whence, $a = 5 \text{ or } -15$

Therefore the quantities are

5, 10, 20, and 40; or, -15, 30, -60, and 120.

4. Let the quantities be a, ar, and ar^2; then by the conditions,
$$\begin{cases} a + ar = 20 & (1) \\ ar^2 - ar = 30 & (2) \end{cases}$$

Dividing (2) by (1), $\dfrac{r^2 - r}{1 + r} = \dfrac{3}{2}$

$$2r^2 - 2r = 3 + 3r$$
$$2r^2 - 5r = 3$$
$$16r^2 - 40r + 25 = 24 + 25 = 49$$
$$4r - 5 = \pm 7$$
$$4r = 12 \text{ or } -2$$
$$r = 3 \text{ or } -\frac{1}{2}$$

Whence by (1), $a = \dfrac{20}{1 + r}$

$= 5 \text{ or } 40$

Therefore the quantities are

5, 15, and 45; or, 40, -20, and 10.

5.
$$\begin{cases} ar^3 = -108 & (1) \\ ar^7 = -8748 & (2) \end{cases}$$

Dividing (2) by (1), $r^4 = 81$

Whence, $r = \pm 3$

Substituting in (1), $\pm 27a = -108$

Whence, $a = \mp 4$

6. $$r=\frac{3}{2},\ n=7,\ S=2059$$

By (I.), $$l=a\left(\frac{3}{2}\right)^6 \qquad (1)$$

By (II.), $$2059=\frac{\frac{3}{2}l-a}{\frac{3}{2}-1} \qquad (2)$$

From (2), $$3l-2a=2059 \qquad (3)$$

From (1), $$l=\frac{729a}{64} \qquad (4)$$

Substituting in (3), $$\frac{2187a}{64}-2a=2059$$

$$\frac{2059a}{64}=2059$$

Whence, $$a=64$$

7. To find the sum to infinity of

$$100+200(.9375)+200(.9375)^2+\cdots$$

$$a=200(.9375)=187.5$$

$$r=.9375$$

$$\therefore\ S=\frac{187.5}{1-.9375}=\frac{187.5}{.0625}=3000$$

$$3000+100=3100$$

8. Let the quantities be a, ar, ar^2, and ar^3; then by the conditions,

$$\begin{cases} a+ar+ar^2+ar^3=30 & (1)\\ \dfrac{ar^3}{ar+ar^2}=\dfrac{4}{3} & (2)\end{cases}$$

From (2), $$\frac{r^2}{1+r}=\frac{4}{3}$$

$$3r^2-4r=4$$

$$9r^2-12r+4=16$$

$$3r-2=\pm 4$$

$$3r=6 \text{ or } -2$$

$$r=2 \text{ or } -\frac{2}{3}$$

Whence by (1), $$a+2a+4a+8a=30$$

or, $$a-\frac{2a}{3}+\frac{4a}{9}-\frac{8a}{27}=30$$

That is, $$15a=30$$

and $$\frac{13a}{27}=30$$

Whence, $$a = 2 \text{ or } \frac{810}{13}$$

Therefore the quantities are

$$2,\ 4,\ 8,\ \text{and } 16;\ \text{or, } \frac{810}{13},\ -\frac{540}{13},\ \frac{360}{13},\ \text{and } -\frac{240}{13}.$$

9. $$\begin{cases} ar^2 = \dfrac{1}{24} & (1) \\ ar^5 = \dfrac{9}{512} & (2) \end{cases}$$

Dividing (2) by (1), $$r^3 = \frac{27}{64}$$

$$r = \frac{3}{4}$$

Whence, $$r^2 = \frac{9}{16} \quad (3)$$

Multiplying (2) by (3), $$ar^7 = \frac{81}{8192}$$

Therefore the eighth term is $\frac{81}{8192}$.

10. Let the parts be a, ar, and ar^2; then by the conditions,

$$\begin{cases} a + ar + ar^2 = 39 & (1) \\ ar^2 - a = 24 & (2) \end{cases}$$

Dividing (2) by (1), $$\frac{r^2 - 1}{1 + r + r^2} = \frac{8}{13}$$

$$13r^2 - 13 = 8 + 8r + 8r^2$$

$$5r^2 - 8r = 21$$

$$25r^2 - 40r + 16 = 105 + 16 = 121$$

$$5r - 4 = \pm 11$$

$$5r = 15$$

$$r = 3$$

Whence by (2), $$a = \frac{24}{r^2 - 1} = 3$$

Therefore the parts are 3, 9, and 27.

11. Let the numbers be a, ar, and ar^2; then by the conditions,

$$\begin{cases} a^3r^3 = 64 & (1) \\ a^2 + a^2r^4 = 68 & (2) \end{cases}$$

From (1), $$ar = 4 \quad (3)$$

Or, $$a^2r^2 = 16 \quad (4)$$

Dividing (2) by (4), $\frac{1+r^4}{r^2}=\frac{17}{4}$

$$4r^4-17r^2=-4$$

$$64r^4-272r^2+289=-64+289=225$$

$$8r^2-17=\pm 15$$

$$8r^2=32 \text{ or } 2$$

$$r^2=4 \text{ or } \frac{1}{4}$$

$$r=\pm 2 \text{ or } \pm\frac{1}{2}$$

Substituting in (3), $a=\frac{4}{\pm 2} \text{ or } \frac{4}{\pm\frac{1}{2}}$

$$=\pm 2 \text{ or } \pm 8$$

Therefore the numbers are

2, 4, and 8; or, -2, 4, and -8.

12. Let the quantities be a, ar, and ar^2; then by the conditions,

$$\begin{cases} a^3r^3=8 & (1) \\ a^3+a^3r^3+a^3r^6=73 & (2) \end{cases}$$

From (1), $ar=2$ (3)

Dividing (2) by (1), $\frac{1+r^3+r^6}{r^3}=\frac{73}{8}$

$$8+8r^3+8r^6=73r^3$$

$$8r^6-65r^3=-8$$

$$256r^6-32\times 65r^3+65^2=3969$$

$$16r^3-65=\pm 63$$

$$16r^3=128 \text{ or } 2$$

$$r^3=8 \text{ or } \frac{1}{8}$$

$$r=2 \text{ or } \frac{1}{2}$$

Whence by (3), $a=\frac{2}{r}=1 \text{ or } 4$

Therefore the quantities are 1, 2, and 4.

CHAPTER XXIX.

Art. 332. — Pages 284, 285.

2. $(c^{\frac{2}{3}} + d^{-\frac{3}{4}})^4 = (c^{\frac{2}{3}})^4 + 4(c^{\frac{2}{3}})^3 d^{-\frac{3}{4}} + 6(c^{\frac{2}{3}})^2(d^{-\frac{3}{4}})^2$
$+ 4c^{\frac{2}{3}}(d^{-\frac{3}{4}})^3 + (d^{-\frac{3}{4}})^4$
$= c^{\frac{8}{3}} + 4c^2d^{-\frac{3}{4}} + 6c^{\frac{4}{3}}d^{-\frac{3}{2}} + 4c^{\frac{2}{3}}d^{-\frac{9}{4}} + d^{-3}.$

3. $(m^{-\frac{1}{2}} - n^2)^5 = (m^{-\frac{1}{2}})^5 - 5(m^{-\frac{1}{2}})^4n^2 + 10(m^{-\frac{1}{2}})^3(n^2)^2$
$- 10(m^{-\frac{1}{2}})^2(n^2)^3 + 5m^{-\frac{1}{2}}(n^2)^4 - (n^2)^5$
$= m^{-\frac{5}{2}} - 5m^{-2}n^2 + 10m^{-\frac{3}{2}}n^4 - 10m^{-1}n^6 + 5m^{-\frac{1}{2}}n^8 - n^{10}.$

4. $\left(\frac{x}{y} - \frac{3y}{x}\right)^3 = (xy^{-1} - 3x^{-1}y)^3$
$= (xy^{-1})^3 - 3(xy^{-1})^2(3x^{-1}y) + 3(xy^{-1})(3x^{-1}y)^2 - (3x^{-1}y)^3$
$= x^3y^{-3} - 9xy^{-1} + 27x^{-1}y - 27x^{-3}y^3.$

5. $(x^m + 2y^n)^5 = (x^m)^5 + 5(x^m)^4(2y^n) + 10(x^m)^3(2y^n)^2$
$+ 10(x^m)^2(2y^n)^3 + 5x^m(2y^n)^4 + (2y^n)^5$
$= x^{5m} + 10x^{4m}y^n + 40x^{3m}y^{2n} + 80x^{2m}y^{3n} + 80x^my^{4n} + 32y^{5n}.$

6. $(a^3 + 3\sqrt{x})^4 = (a^3 + 3x^{\frac{1}{2}})^4$
$= (a^3)^4 + 4(a^3)^3(3x^{\frac{1}{2}}) + 6(a^3)^2(3x^{\frac{1}{2}})^2 + 4a^3(3x^{\frac{1}{2}})^3 + (3x^{\frac{1}{2}})^4$
$= a^{12} + 12a^9x^{\frac{1}{2}} + 54a^6x + 108a^3x^{\frac{3}{2}} + 81x^2.$

7. $\left(\frac{m}{n} - \sqrt{mn}\right)^5 = (mn^{-1} - m^{\frac{1}{2}}n^{\frac{1}{2}})^5$
$= (mn^{-1})^5 - 5(mn^{-1})^4m^{\frac{1}{2}}n^{\frac{1}{2}} + 10(mn^{-1})^3(m^{\frac{1}{2}}n^{\frac{1}{2}})^2$
$- 10(mn^{-1})^2(m^{\frac{1}{2}}n^{\frac{1}{2}})^3 + 5(mn^{-1})(m^{\frac{1}{2}}n^{\frac{1}{2}})^4 - (m^{\frac{1}{2}}n^{\frac{1}{2}})^5$
$= m^5n^{-5} - 5m^{\frac{9}{2}}n^{-\frac{7}{2}} + 10m^4n^{-2} - 10m^{\frac{7}{2}}n^{-\frac{1}{2}} + 5m^3n - m^{\frac{5}{2}}n^{\frac{5}{2}}.$

8. $\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{y^2}}+\dfrac{\sqrt[3]{y}}{\sqrt[3]{x^2}}\right)^3=(x^{\frac{1}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{1}{3}})^3$

$=(x^{\frac{1}{3}}y^{-\frac{2}{3}})^3+3(x^{\frac{1}{3}}y^{-\frac{2}{3}})^2x^{-\frac{2}{3}}y^{\frac{1}{3}}+3x^{\frac{1}{3}}y^{-\frac{2}{3}}(x^{-\frac{2}{3}}y^{\frac{1}{3}})^2+(x^{-\frac{2}{3}}y^{\frac{1}{3}})^3$

$=xy^{-2}+3y^{-1}+3x^{-1}+x^{-2}y.$

9. $\left(m^2-\dfrac{n^3}{2}\right)^4=(m^2)^4-4(m^2)^3\dfrac{n^3}{2}+6(m^2)^2\left(\dfrac{n^3}{2}\right)^2-4m^2\left(\dfrac{n^3}{2}\right)^3+\left(\dfrac{n^3}{2}\right)^4$

$=m^8-2m^6n^3+\dfrac{3m^4n^6}{2}-\dfrac{m^2n^9}{2}+\dfrac{n^{12}}{16}.$

10. $(a^{\frac{1}{2}}b^{-\frac{1}{3}}-a^{-\frac{1}{2}}b^{\frac{1}{3}})^5=(a^{\frac{1}{2}}b^{-\frac{1}{3}})^5-5(a^{\frac{1}{2}}b^{-\frac{1}{3}})^4a^{-\frac{1}{2}}b^{\frac{1}{3}}$

$+10(a^{\frac{1}{2}}b^{-\frac{1}{3}})^3(a^{-\frac{1}{2}}b^{\frac{1}{3}})^2-10(a^{\frac{1}{2}}b^{-\frac{1}{3}})^2(a^{-\frac{1}{2}}b^{\frac{1}{3}})^3$

$+5a^{\frac{1}{2}}b^{-\frac{1}{3}}(a^{-\frac{1}{2}}b^{\frac{1}{3}})^4-(a^{-\frac{1}{2}}b^{\frac{1}{3}})^5$

$=a^{\frac{5}{2}}b^{-\frac{5}{3}}-5a^{\frac{3}{2}}b^{-1}+10a^{\frac{1}{2}}b^{-\frac{1}{3}}-10a^{-\frac{1}{2}}b^{\frac{1}{3}}+5a^{-\frac{3}{2}}b-a^{-\frac{5}{2}}b^{\frac{5}{3}}.$

11. $(\sqrt{a^3}-3\sqrt[3]{a})^4=(a^{\frac{3}{2}}-3a^{\frac{1}{3}})^4$

$=(a^{\frac{3}{2}})^4-4(a^{\frac{3}{2}})^3(3a^{\frac{1}{3}})+6(a^{\frac{3}{2}})^2(3a^{\frac{1}{3}})^2-4a^{\frac{3}{2}}(3a^{\frac{1}{3}})^3+(3a^{\frac{1}{3}})^4$

$=a^6-12a^{\frac{29}{6}}+54a^{\frac{11}{3}}-108a^{\frac{5}{2}}+81a^{\frac{4}{3}}.$

12. $\left(\dfrac{2x}{\sqrt{y}}+\dfrac{y}{2\sqrt{x}}\right)^4=\left(2xy^{-\frac{1}{2}}+\dfrac{x^{-\frac{1}{2}}y}{2}\right)^4$

$=(2xy^{-\frac{1}{2}})^4+4(2xy^{-\frac{1}{2}})^3\dfrac{x^{-\frac{1}{2}}y}{2}+6(2xy^{-\frac{1}{2}})^2\left(\dfrac{x^{-\frac{1}{2}}y}{2}\right)^2$

$+4(2xy^{-\frac{1}{2}})\left(\dfrac{x^{-\frac{1}{2}}y}{2}\right)^3+\left(\dfrac{x^{-\frac{1}{2}}y}{2}\right)^4$

$=16x^4y^{-2}+16x^{\frac{5}{2}}y^{-\frac{1}{2}}+6xy+x^{-\frac{1}{2}}y^{\frac{5}{2}}+\dfrac{x^{-2}y^4}{16}.$

13. $\left(a^{-2}-\dfrac{1}{3}x^{\frac{1}{2}}\right)^6=(a^{-2})^6-6(a^{-2})^5\left(\dfrac{1}{3}x^{\frac{1}{2}}\right)$

$+15(a^{-2})^4\left(\dfrac{1}{3}x^{\frac{1}{2}}\right)^2-20(a^{-2})^3\left(\dfrac{1}{3}x^{\frac{1}{2}}\right)^3$

$+15(a^{-2})^2\left(\dfrac{1}{3}x^{\frac{1}{2}}\right)^4-6a^{-2}\left(\dfrac{1}{3}x^{\frac{1}{2}}\right)^5+\left(\dfrac{1}{3}x^{\frac{1}{2}}\right)^6$

$=a^{-12}-2a^{-10}x^{\frac{1}{2}}+\dfrac{5a^{-8}x}{3}-\dfrac{20a^{-6}x^{\frac{3}{2}}}{27}+\dfrac{5a^{-4}x^2}{27}-\dfrac{2a^{-2}x^{\frac{5}{2}}}{81}+\dfrac{x^3}{729}.$

14. $(x^{\frac{3}{5}}+3y^{-\frac{2}{5}})^5=(x^{\frac{3}{5}})^5+5(x^{\frac{3}{5}})^4(3y^{-\frac{2}{5}})+10(x^{\frac{3}{5}})^3(3y^{-\frac{2}{5}})^2$
$+10(x^{\frac{3}{5}})^2(3y^{-\frac{2}{5}})^3+5x^{\frac{3}{5}}(3y^{-\frac{2}{5}})^4+(3y^{-\frac{2}{5}})^5$
$=x^3+15x^{\frac{12}{5}}y^{-\frac{2}{5}}+90x^{\frac{9}{5}}y^{-\frac{4}{5}}+270x^{\frac{6}{5}}y^{-\frac{6}{5}}+405x^{\frac{3}{5}}y^{-\frac{8}{5}}+243y^{-2}.$

15. $$\left(\frac{a\sqrt{b}}{2x^{\frac{1}{3}}}-\frac{2\sqrt[3]{x}}{a^2b^{\frac{1}{2}}}\right)^3=\left(\frac{ab^{\frac{1}{2}}x^{-\frac{1}{3}}}{2}-2a^{-2}b^{-\frac{1}{2}}x^{\frac{1}{3}}\right)^3$$
$$=\left(\frac{ab^{\frac{1}{2}}x^{-\frac{1}{3}}}{2}\right)^3-3\left(\frac{ab^{\frac{1}{2}}x^{-\frac{1}{3}}}{2}\right)^2(2a^{-2}b^{-\frac{1}{2}}x^{\frac{1}{3}})$$
$$+3\left(\frac{ab^{\frac{1}{2}}x^{-\frac{1}{3}}}{2}\right)(2a^{-2}b^{-\frac{1}{2}}x^{\frac{1}{3}})^2-(2a^{-2}b^{-\frac{1}{2}}x^{\frac{1}{3}})^3$$
$$=\frac{a^3b^{\frac{3}{2}}x^{-1}}{8}-\frac{3b^{\frac{1}{2}}x^{-\frac{1}{3}}}{2}+6a^{-3}b^{-\frac{1}{2}}x^{\frac{1}{3}}-8a^{-6}b^{-\frac{3}{2}}x.$$

16. $(3a^{-\frac{3}{4}}\sqrt{b}-b^{-\frac{1}{2}}\sqrt[4]{a})^4=(3a^{-\frac{3}{4}}b^{\frac{1}{2}}-a^{\frac{1}{4}}b^{-\frac{1}{2}})^4$
$=(3a^{-\frac{3}{4}}b^{\frac{1}{2}})^4-4(3a^{-\frac{3}{4}}b^{\frac{1}{2}})^3a^{\frac{1}{4}}b^{-\frac{1}{2}}+6(3a^{-\frac{3}{4}}b^{\frac{1}{2}})^2(a^{\frac{1}{4}}b^{-\frac{1}{2}})^2$
$-4(3a^{-\frac{3}{4}}b^{\frac{1}{2}})(a^{\frac{1}{4}}b^{-\frac{1}{2}})^3+(a^{\frac{1}{4}}b^{-\frac{1}{2}})^4$
$=81a^{-3}b^2-108a^{-2}b+54a^{-1}-12b^{-1}+ab^{-2}.$

17. $$\left(\sqrt{\frac{a}{b}}+2\sqrt{\frac{b}{a}}\right)^6=(a^{\frac{1}{2}}b^{-\frac{1}{2}}+2a^{-\frac{1}{2}}b^{\frac{1}{2}})^6$$
$=(a^{\frac{1}{2}}b^{-\frac{1}{2}})^6+6(a^{\frac{1}{2}}b^{-\frac{1}{2}})^5(2a^{-\frac{1}{2}}b^{\frac{1}{2}})+15(a^{\frac{1}{2}}b^{-\frac{1}{2}})^4(2a^{-\frac{1}{2}}b^{\frac{1}{2}})^2$
$+20(a^{\frac{1}{2}}b^{-\frac{1}{2}})^3(2a^{-\frac{1}{2}}b^{\frac{1}{2}})^3+15(a^{\frac{1}{2}}b^{-\frac{1}{2}})^2(2a^{-\frac{1}{2}}b^{\frac{1}{2}})^4$
$+6(a^{\frac{1}{2}}b^{-\frac{1}{2}})(2a^{-\frac{1}{2}}b^{\frac{1}{2}})^5+(2a^{-\frac{1}{2}}b^{\frac{1}{2}})^6$
$=a^3b^{-3}+12a^2b^{-2}+60ab^{-1}+160+240a^{-1}b+192a^{-2}b^2+64a^{-3}b^3$
$$=\frac{a^3}{b^3}+\frac{12a^2}{b^2}+\frac{60a}{b}+160+\frac{240b}{a}+\frac{192b^2}{a^2}+\frac{64b^3}{a^3}.$$

18. $(1-x-x^2)^4=[(1-x)-x^2]^4$
$=(1-x)^4-4(1-x)^3x^2+6(1-x)^2x^4-4(1-x)x^6+x^8$
$=1-4x+6x^2-4x^3+x^4-4x^2+12x^3-12x^4+4x^5$
$+6x^4-12x^5+6x^6-4x^6+4x^7+x^8$
$=1-4x+2x^2+8x^3-5x^4-8x^5+2x^6+4x^7+x^8.$

19. $(x^2+x-1)^4=[x^2+(x-1)]^4$
$=x^8+4x^6(x-1)+6x^4(x-1)^2+4x^2(x-1)^3+(x-1)^4$
$=x^8+4x^7-4x^6+6x^6-12x^5+6x^4+4x^5-12x^4+12x^3$
$-4x^2+x^4-4x^3+6x^2-4x+1$
$=x^8+4x^7+2x^6-8x^5-5x^4+8x^3+2x^2-4x+1.$

20. $(1+2x-x^2)^4=[(1+2x)-x^2]^4$
$=(1+2x)^4-4(1+2x)^3x^2+6(1+2x)^2x^4-4(1+2x)x^6+x^8$
$=1+8x+24x^2+32x^3+16x^4-4x^2-24x^3-48x^4-32x^5$
$+6x^4+24x^5+24x^6-4x^6-8x^7+x^8$
$=1+8x+20x^2+8x^3-26x^4-8x^5+20x^6-8x^7+x^8.$

21. $(1-x+x^2)^5=[(1-x)+x^2]^5$
$=(1-x)^5+5(1-x)^4x^2+10(1-x)^3x^4+10(1-x)^2x^6+5$
$(1-x)x^8+x^{10}$
$=1-5x+10x^2-10x^3+5x^4-x^5+5x^2-20x^3+30x^4$
$-20x^5+5x^6+10x^4-30x^5+30x^6-10x^7+10x^6$
$-20x^7+10x^8+5x^8-5x^9+x^{10}$
$=1-5x+15x^2-30x^3+45x^4-51x^5+45x^6-30x^7+15x^8$
$-5x^9+x^{10}.$

Art. 333.—Page 286.

2. $r=7$, $n=11$. Hence the result
$$=\frac{11\cdot10\cdot9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4\cdot5\cdot6}a^5x^6=462a^5x^6.$$

3. $r=6$, $n=10$. Hence the result
$$=\frac{10\cdot9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4\cdot5}m^5=252m^5.$$

4. $r=8$, $n=12$. Hence the result
$$=\frac{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}c^5(-d)^7=-792c^5d^7.$$

5. $r=5$, $n=14$. Hence the result
$$=\frac{14\cdot13\cdot12\cdot11}{1\cdot2\cdot3\cdot4}(-a^2)^4=1001a^8.$$

6. $r=7$, $n=9$. Hence the result
$$=\frac{9\cdot8\cdot7\cdot6\cdot5\cdot4}{1\cdot2\cdot3\cdot4\cdot5\cdot6}\left(\frac{a}{b}\right)^3\left(\frac{b}{a}\right)^6=\frac{84b^3}{a^3}.$$

7. $r=5$, $n=13$. Hence the result

$$=\frac{13\cdot12\cdot11\cdot10}{1\cdot2\cdot3\cdot4}x^9(-x^{\frac{1}{2}})^4=715x^{11}.$$

8. $r=6$, $n=9$. Hence the result

$$=\frac{9\cdot8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4\cdot5}(a^{-3})^4\left(-\frac{1}{2}ab\right)^5=-\frac{63a^{-7}b^5}{16}$$

9. $r=8$, $n=10$. Hence the result

$$=\frac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}(x^{-1})^3(2y^{\frac{1}{2}})^7=15360x^{-3}y^{\frac{7}{2}}.$$

10. $r=4$, $n=11$. Hence the result

$$=\frac{11\cdot10\cdot9}{1\cdot2\cdot3}(a^{\frac{2}{3}})^8(-3x^{-1})^3=-4455a^{\frac{16}{3}}x^{-3}.$$

11. $r=9$, $n=12$. Hence the result

$$=\frac{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}(m^{\frac{1}{2}})^4(2m^{-\frac{1}{4}})^8=126720.$$

CHAPTER XXX.

Art. 343.—Page 290.

2. $\log\ \ 6 = \log(2 \times 3) = \log 2 + \log 3 = .3010 + .4771 = .7781.$

3. $\log\ 14 = \log(2 \times 7) = \log 2 + \log 7 = .3010 + .8451 = 1.1461.$

4. $\log\ \ 8 = \log(2 \times 2 \times 2) = \log 2 + \log 2 + \log 2 = 3 \log 2$
$= 3 \times .3010 = .9030.$

5. $\log\ 12 = \log(2 \times 2 \times 3) = \log 2 + \log 2 + \log 3$
$= 2 \log 2 + \log 3 = .6020 + .4771 = 1.0791.$

6. $\log\ 15 = \log(3 \times 5) = \log 3 + \log 5 = .4771 + .6990 = 1.1761.$

7. $\log\ 21 = \log(3 \times 7) = \log 3 + \log 7 = .4771 + .8451 = 1.3222.$

8. $\log\ 63 = \log(3 \times 3 \times 7) = \log 3 + \log 3 + \log 7 = 2 \log 3 + \log 7$
$= .9542 + .8451 = 1.7993.$

9. $\log\ 56 = \log(2 \times 2 \times 2 \times 7) = 3 \log 2 + \log 7$
$= .9030 + .8451 = 1.7481.$

10. $\log\ 84 = \log(2 \times 2 \times 3 \times 7) = 2 \log 2 + \log 3 + \log 7$
$= .6020 + .4771 + .8451 = 1.9242.$

11. $\log\ 45 = \log(3 \times 3 \times 5) = 2 \log 3 + \log 5 = .9542 + .6990 = 1.6532.$

12. $\log\ 98 = \log(2 \times 7 \times 7) = \log 2 + 2 \log 7 = .3010 + 1.6902 = 1.9912.$

13. $\log 105 = \log(3 \times 5 \times 7) = \log 3 + \log 5 + \log 7$
$= .4771 + .6990 + .8451 = 2.0212.$

14. $\log 112 = \log(2 \times 2 \times 2 \times 2 \times 7) = 4 \log 2 + \log 7$
$= 1.2040 + .8451 = 2.0491.$

15. $\log 144 = \log(2 \times 2 \times 2 \times 2 \times 3 \times 3) = 4 \log 2 + 2 \log 3$
$= 1.2040 + .9542 = 2.1582.$

16. $\log 216 = \log(2 \times 2 \times 2 \times 3 \times 3 \times 3) = 3 \log 2 + 3 \log 3$
$= .9030 + 1.4313 = 2.3343.$

17. $\log 135 = \log(3 \times 3 \times 3 \times 5) = 3\log 3 + \log 5$
$= 1.4313 + .6990 = 2.1303.$

18. $\log 168 = \log(2 \times 2 \times 2 \times 3 \times 7) = 3\log 2 + \log 3 + \log 7$
$= .9030 + .4771 + .8451 = 2.2252.$

19. $\log 147 = \log(3 \times 7 \times 7) = \log 3 + 2\log 7 = .4771 + 1.6902 = 2.1673$

20. $\log 375 = \log(3 \times 5 \times 5 \times 5) = \log 3 + 3\log 5$
$= .4771 + 2.0970 = 2.5741.$

21. $\log 343 = \log(7 \times 7 \times 7) = 3\log 7 = 2.5353.$

Art. 345. — Page 291.

2. $\log \frac{7}{3} = \log 7 - \log 3 = .8451 - .4771 = .3680.$

3. $\log \frac{10}{7} = \log 10 - \log 7 = 1 - .8451 = .1549.$

4. $\log 3\frac{1}{3} = \log \frac{10}{3} = \log 10 - \log 3 = 1 - .4771 = .5229.$

5. $\log 35 = \log \frac{70}{2} = \log(10 \times 7) - \log 2 = \log 10 + \log 7 - \log 2$
$= 1 + .8451 - .3010 = 1.5441.$

6. $\log \frac{21}{16} = \log 21 - \log 16 = \log(3 \times 7) - \log(2 \times 2 \times 2 \times 2)$
$= \log 3 + \log 7 - 4\log 2 = .4771 + .8451 - 1.2040 = .1182.$

7. $\log 125 = \log(5 \times 5 \times 5) = 3\log 5.$
$\log 5 = \log \frac{10}{2} = 1 - \log 2 = 1 - .3010 = .6990.$
$\therefore \log 125 = 3 \times .6990 = 2.0970.$

8. $\log \frac{42}{25} = \log(2 \times 3 \times 7) - \log(5 \times 5) = \log 2 + \log 3 + \log 7 - 2\log 5$
$\log 5 = \log \frac{10}{2} = 1 - \log 2 = .6990.$
$\therefore \log \frac{42}{25} = .3010 + .4771 + .8451 - 1.3980 = .2252.$

9. $\log 175 = \log(5 \times 5 \times 7) = 2\log 5 + \log 7.$
$\log 5 = \log \frac{10}{2} = 1 - \log 2 = .6990.$
$\therefore \log 175 = 1.3980 + .8451 = 2.2431.$

10. $\log 11\frac{1}{9} = \log \frac{100}{9} = \log 100 - \log (3 \times 3)$

$= 2 - 2 \log 3 = 2 - .9542 = 1.0458.$

11. $\log\ 7\frac{1}{7} = \log \frac{50}{7} = \log \frac{100}{14} = \log 100 - \log (2 \times 7)$

$= 2 - \log 2 - \log 7 = 2 - .3010 - .8451 = .8539.$

12. $\log\ \frac{35}{6} = \log (5 \times 7) - \log (2 \times 3) = \log 5 + \log 7 - \log 2 - \log 3.$

$\log\ 5 = \log \frac{10}{2} = 1 - \log 2 = .6990.$

$\therefore \log\ \frac{35}{6} = .6990 + .8451 - .3010 - .4771 = .7660.$

13. $\log\ 5\frac{4}{9} = \log \frac{49}{9} = \log (7 \times 7) - \log (3 \times 3) = 2 \log 7 - 2 \log 3$

$= 1.6902 - .9542 = .7360.$

Art. 348. — Page 292.

3. $\log\ 3^{\frac{3}{5}} = \frac{3}{5} \log 3 = \frac{3}{5} \times .4771 = .2863.$

4. $\log\ 2^9 = 9 \log 2 = 9 \times .3010 = 2.7090.$

5. $\log\ 7^5 = 5 \log 7 = 5 \times .8451 = 4.2255.$

6. $\log\ 5^{\frac{1}{5}} = \frac{1}{5} \log 5.$

$\log\ 5 = \log \frac{10}{2} = 1 - \log 2 = .6990.$

$\therefore \log\ 5^{\frac{1}{5}} = \frac{.6990}{5} = .1398.$

7. $\log\ 12^{\frac{2}{3}} = \frac{2}{3} \log 12.$

$\log\ 12 = \log (2 \times 2 \times 3) = 2 \log 2 + \log 3 = .6020 + .4771 = 1.0791$

$\therefore \log\ 12^{\frac{2}{3}} = \frac{2}{3} \times 1.0791 = .7194.$

8. $\log\ 21^{\frac{1}{2}} = \frac{1}{2} \log 21.$

$\log\ 21 = \log (3 \times 7) = \log 3 + \log 7 = .4771 + .8451 = 1.3222.$

$\therefore \log\ 21^{\frac{1}{2}} = \frac{1.3222}{2} = .6611.$

9. $\log 14^4 = 4 \log 14.$

$\log 14 = \log(2 \times 7) = \log 2 + \log 7 = .3010 + .8451 = 1.1461.$

$\therefore \log 14^4 = 4 \times 1.1461 = 4.5844.$

10. $\log 25^{\frac{7}{3}} = \frac{7}{3} \log 25.$

$\log 25 = \log \frac{100}{4} = \log 100 - \log(2 \times 2)$

$= 2 - 2 \log 2 = 2 - .6020 = 1.3980.$

$\therefore \log 25^{\frac{7}{3}} = \frac{7}{3} \times 1.3980 = 3.2620.$

11. $\log 15^{\frac{5}{6}} = \frac{5}{6} \log 15.$

$\log 15 = \log(3 \times 5) = \log 3 + \log 5.$

$\log 5 = \log \frac{10}{2} = 1 - \log 2 = .6990.$

$\therefore \log 15 = .4771 + .6990 = 1.1761.$

$\therefore \log 15^{\frac{5}{6}} = \frac{5}{6} \times 1.1761 = .9801.$

12. $\log \sqrt{7} = \frac{\log 7}{2} = \frac{.8451}{2} = .4225.$

13. $\log \sqrt[3]{3} = \frac{\log 3}{3} = \frac{.4771}{3} = .1590.$

14. $\log \sqrt[7]{2} = \frac{\log 2}{7} = \frac{.3010}{7} = .0430.$

15. $\log \sqrt[6]{5} = \frac{\log 5}{6}.$

$\log 5 = \log \frac{10}{2} = 1 - \log 2 = .6990.$

$\therefore \log \sqrt[6]{5} = \frac{.6990}{6} = .1165.$

16. $\log \sqrt[4]{35} = \frac{\log 35}{4}.$

$\log 35 = \log(5 \times 7) = \log 5 + \log 7.$

$\log 5 = \log \frac{10}{2} = 1 - \log 2 = .6990.$

$\therefore \log 35 = .6990 + .8451 = 1.5441.$

$\therefore \log \sqrt[4]{35} = \frac{1.5441}{4} = .3860.$

17. $\log \sqrt[9]{98} = \dfrac{\log 98}{9}.$

$\log \quad 98 = \log(2 \times 7^2) = \log 2 + 2\log 7 = .3010 + 1.6902 = 1.9912.$

$\therefore \log \sqrt[9]{98} = \dfrac{1.9912}{9} = .2212.$

18. $\log \sqrt[12]{126} = \dfrac{\log 126}{12}.$

$\log \quad 126 = \log(2 \times 3^2 \times 7) = \log 2 + 2\log 3 + \log 7$
$= .3010 + .9542 + .8451 = 2.1003.$

$\therefore \log \sqrt[12]{126} = \dfrac{2.1003}{12} = .1750.$

20. $\log\left(\dfrac{10}{3}\right)^5 = 5\log\dfrac{10}{3} = 5(\log 10 - \log 3) = 5(1 - .4771)$
$= 5 \times .5229 = 2.6145.$

21. $\log \dfrac{7^{\frac{3}{4}}}{5^{\frac{2}{3}}} = \log 7^{\frac{3}{4}} - \log 5^{\frac{2}{3}} = \dfrac{3}{4}\log 7 - \dfrac{2}{3}\log 5$
$= .6338 - .4660 = .1678.$

22. $\log(3^{\frac{1}{6}} \times 2^{\frac{3}{5}}) = \log 3^{\frac{1}{6}} + \log 2^{\frac{3}{5}} = \dfrac{1}{6}\log 3 + \dfrac{3}{5}\log 2$
$= .0795 + .1806 = .2601.$

23. $\log 3\sqrt[4]{7} = \log 3 + \log \sqrt[4]{7} = \log 3 + \dfrac{\log 7}{4}$
$= .4771 + .2113 = .6884.$

24. $\log \sqrt{\dfrac{7}{3}} = \dfrac{1}{2}\log\dfrac{7}{3} = \dfrac{\log 7 - \log 3}{2} = \dfrac{.8451 - .4771}{2} = .1840.$

25. $\log \dfrac{\sqrt[3]{7}}{\sqrt[5]{2}} = \log \sqrt[3]{7} - \log \sqrt[5]{2} = \dfrac{\log 7}{3} - \dfrac{\log 2}{5}$
$= .2817 - .0602 = .2215.$

26. $\log \sqrt[3]{\dfrac{28}{5}} = \dfrac{1}{3}\log\dfrac{28}{5} = \dfrac{\log 28 - \log 5}{3}.$

$\log \quad 28 = \log(2^2 \times 7) = 2\log 2 + \log 7 = .6020 + .8451 = 1.4471$

$\therefore \log \sqrt[3]{\dfrac{28}{5}} = \dfrac{1.4471 - .6990}{3} = .2494.$

27. $\log \dfrac{\sqrt{42}}{10^{\frac{2}{3}}} = \log\sqrt{42} - \log 10^{\frac{2}{3}} = \dfrac{\log 42}{2} - \dfrac{2}{3}\log 10.$

$\log \quad 42 = \log(2 \times 3 \times 7) = \log 2 + \log 3 + \log 7$
$= .3010 + .4771 + .8451 = 1.6232.$

$\therefore \log \dfrac{\sqrt{42}}{10^{\frac{2}{3}}} = .8116 - .6667 = .1449.$

Art. 350. — Page 293.

2. $\log 18 = \log(2 \times 3^2) = \log 2 + 2\log 3 = .3010 + .9542 = 1.2552.$
$\therefore \log 1.8 = 0.2552.$

3. $\log 225 = \log(3^2 \times 5^2) = 2\log 3 + 2\log 5$
$= .9542 + 1.3980 = 2.3522.$
$\therefore \log 2.25 = 0.3522.$

4. $\log 196 = \log(2^2 \times 7^2) = 2\log 2 + 2\log 7$
$= .6020 + 1.6902 = 2.2922.$
$\therefore \log .196 = 9.2922 - 10.$

5. $\log 48 = \log(2^4 \times 3) = 4\log 2 + \log 3 = 1.2040 + .4771 = 1.6811.$
$\therefore \log .048 = 8.6811 - 10.$

6. $\log 384 = \log(2^7 \times 3) = 7\log 2 + \log 3 = 2.1070 + .4771 = 2.5841.$
$\therefore \log 38.4 = 1.5841.$

7. $\log 54 = \log(2 \times 3^3) = \log 2 + 3\log 3 = .3010 + 1.4313 = 1.7323.$
$\therefore \log .0054 = 7.7323 - 10.$

8. $\log 315 = \log(3^2 \times 5 \times 7) = 2\log 3 + \log 5 + \log 7$
$= .9542 + .6990 + .8451 = 2.4983.$
$\therefore \log .000315 = 6.4983 - 10.$

9. $\log 735 = \log(3 \times 5 \times 7^2) = \log 3 + \log 5 + 2\log 7$
$= .4771 + .6990 + 1.6902 = 2.8663.$
$\therefore \log 7350 = 3.8663.$

10. $\log 405 = \log(3^4 \times 5) = 4\log 3 + \log 5$
$= 1.9084 + .6990 = 2.6074.$
$\therefore \log 4.05 = 0.6074.$

11. $\log 448 = \log(2^6 \times 7) = 6\log 2 + \log 7$
$= 1.8060 + .8451 = 2.6511.$
$\therefore \log .448 = 9.6511 - 10.$

12. $\log 3024 = \log(2^4 \times 3^3 \times 7) = 4\log 2 + 3\log 3 + \log 7$
$= 1.2040 + 1.4313 + .8451 = 3.4804.$
$\therefore \log 302.4 = 2.4804.$

13. $\log 6174 = \log(2 \times 3^2 \times 7^3) = \log 2 + 2\log 3 + 3\log 7$
$= .3010 + .9542 + 2.5353 = 3.7905.$
$\therefore \log .06174 = 8.7905 - 10.$

14. $\log\ (8.1)^7 = 7\log 8.1.$

$\log\ 81 = \log 3^4 = 4\log 3 = 1.9084.$

$\therefore \log\ 8.1 = 0.9084.$

$\therefore \log\ (8.1)^7 = 7 \times .9084 = 6.3588.$

15. $\log\ \sqrt[5]{9.6} = \dfrac{\log 9.6}{5}.$

$\log\ 96 = \log(2^5 \times 3) = 5\log 2 + \log 3$
$= 1.5050 + .4771 = 1.9821.$

$\therefore \log\ 9.6 = 0.9821.$

$\therefore \log\ \sqrt[5]{9.6} = \dfrac{0.9821}{5} = .1964.$

16. $\log\ \sqrt[3]{1.62} = \dfrac{\log 1.62}{3}.$

$\log\ 162 = \log(2 \times 3^4) = \log 2 + 4\log 3 = .3010 + 1.9084 = 2.2094.$

$\therefore \log\ 1.62 = 0.2094.$

$\therefore \log\ \sqrt[3]{1.62} = \dfrac{0.2094}{3} = .0698.$

17. $\log(22.4)^{\frac{1}{8}} = \dfrac{1}{8}\log 22.4.$

$\log\ 224 = \log(2^5 \times 7) = 5\log 2 + \log 7 = 1.5050 + .8451 = 2.3501$

$\therefore \log\ 22.4 = 1.3501.$

$\therefore \log(22.4)^{\frac{1}{8}} = \dfrac{1.3501}{8} = .1688.$

Art. 355.—Page 297.

2. $\log\ 80 = 1.9031.$

3. $\log\ 6.3 = 0.7993.$

4. $\log 298 = 2.4742.$

5. $\log .902 = 9.9552 - 10.$

6. Mantissa of $772 = .8876$
$.3 \times 6 = \underline{\quad 2}$
$\therefore \log .7723 = 9.8878 - 10$

7. Mantissa of $105 = .0212$
$.6 \times 41 = \underline{\quad 25}$
$\therefore \log 1056 = 3.0237$

8. Mantissa of $329 = .5172$
$.4 \times 13 = \underline{\quad 5}$
$\therefore \log 3.294 = 0.5177$

9. Mantissa of $520 = .7160$
$.5 \times 8 = \underline{\quad 4}$
$\therefore \log .05205 = 8.7164 - 10$

10. Mantissa of $200 = .3010$
$.8 \times 22 = \underline{\quad 18}$
$\therefore \log 20.08 = 1.3028$

11. Mantissa of $924 = .9657$
$.61 \times 4 = \underline{\quad 2}$
$\therefore \log 92461 = 4.9659$

12. Mantissa of 403 = .6053

$$\begin{array}{r} .22 \times 11 = \underline{\qquad 2} \\ \therefore \log .40322 = 9.6055 - 10 \end{array}$$

13. Mantissa of 717 = .8555

$$\begin{array}{r} .8 \times 6 = \underline{\qquad 5} \\ \therefore \log .007178 = 7.8560 - 10 \end{array}$$

14. Mantissa of 518 = .7143

$$\begin{array}{r} .09 \times 9 = \underline{\qquad 1} \\ \therefore \log 5.1809 = 0.7144 \end{array}$$

15. Mantissa of 103 = .0128

$$\begin{array}{r} .65 \times 42 = \underline{\qquad 27} \\ \therefore \log 1036.5 = 3.0155 \end{array}$$

16. Mantissa of 866 = .9375

$$\begin{array}{r} .76 \times 5 = \underline{\qquad 4} \\ \therefore \log .086676 = 8.9379 - 10 \end{array}$$

17. Mantissa of 115 = .0607

$$\begin{array}{r} .07 \times 38 = \underline{\qquad 3} \\ \therefore \log .11507 = 9.0610 - 10 \end{array}$$

Art. 357. — Page 299.

2. Number corresponding to 1.8055 = 63.9.

3. Number corresponding to 9.4487 − 10 = .281.

4.

$$\begin{array}{lr} 0.2165 & \\ \underline{.2148} = \text{mantissa of } 164 & \\ \frac{17}{27} = & 6 \\ \therefore \text{Number corresponding} = & 1.646 \end{array}$$

5.

$$\begin{array}{lr} 3.9487 & \\ \underline{.9484} = \text{mantissa of } 888 & \\ \frac{3}{5} = & 6 \\ \therefore \text{Number corresponding} = & 8886 \end{array}$$

6. Number corresponding to 2.7364 = 545.

7.

$$\begin{array}{lr} 8.1648 - 10 & \\ \underline{.1644} = \text{mantissa of } 146 & \\ \frac{4}{29} = & 1 \\ \therefore \text{Number corresponding} = & .01461 \end{array}$$

8.

$$\begin{array}{lr} 7.5209 - 10 & \\ \underline{.5198} = \text{mantissa of } 331 & \\ \frac{11}{13} = & 8 \\ \therefore \text{Number corresponding} = & .003318 \end{array}$$

9.

$$\begin{array}{lr} 4.0095 & \\ \underline{.0086} = \text{mantissa of } 102 & \\ \frac{9}{42} = & 21 \\ \therefore \text{Number corresponding} = & 10221 \end{array}$$

10.

$$\begin{array}{lr} 0.9774 & \\ \underline{.9773} = \text{mantissa of } 949 & \\ \frac{1}{4} = & 2 \\ \therefore \text{Number corresponding} = & 9.492 \end{array}$$

11.

$$\begin{array}{lr} 9.3178 - 10 & \\ \underline{.3160} = \text{mantissa of } 207 & \\ \frac{18}{21} = & 9 \\ \therefore \text{Number corresponding} = & .2079 \end{array}$$

12.

$$\begin{array}{lr} 1.6482 & \\ \underline{.6474} = \text{mantissa of } 444 & \\ \frac{8}{10} = & 8 \\ \therefore \text{Number corresponding} = & 44.48 \end{array}$$

13.

$$\begin{array}{lr} 7.0450 - 10 & \\ \underline{.0414} = \text{mantissa of } 110 & \\ \frac{36}{39} = & 9 \\ \therefore \text{Number corresponding} = & .001109 \end{array}$$

14.

$$\begin{array}{lr} 4.8016 & \\ \underline{.8014} = \text{mantissa of } 633 & \\ \frac{2}{7} = & 29 \\ \therefore \text{Number corresponding} = & 63329 \end{array}$$

15. $8.1144 - 10$

$\underline{.1139} =$ mantissa of 130

$\frac{5}{34} =$ 1

$\therefore$ Number corresponding $= \overline{.01301}$

16. 2.7015

$\underline{.7007} =$ mantissa of 502

$\frac{8}{9} =$ 9

$\therefore$ Number corresponding $= \overline{502.9}$

Art. 361.—Pages 302–304.

1. $\log (9.238 \times .9152) = \log 9.238 + \log .9152.$

$$\begin{aligned} \log 9.238 &= 0.9656 \\ \log .9152 &= \underline{9.9615 - 10} \\ &\ 0.9271 = \log 8.454. \end{aligned}$$

2. $\log (130.36 \times .08237) = \log 130.36 + \log .08237.$

$$\begin{aligned} \log 130.36 &= 2.1151 \\ \log .08237 &= \underline{8.9157 - 10} \\ &\ 1.0308 = \log 10.73. \end{aligned}$$

3. $\log (721.3 \times 3.0528) = \log 721.3 + \log 3.0528.$

$$\begin{aligned} \log 721.3 &= 2.8581 \\ \log 3.0528 &= \underline{0.4847} \\ &\ 3.3428 = \log 2202. \end{aligned}$$

Result, -2202.

4. $\log (4.3264 \times .050377) = \log 4.3264 + \log .050377.$

$$\begin{aligned} \log 4.3264 &= 0.6361 \\ \log .050377 &= \underline{8.7022 - 10} \\ &\ 9.3383 - 10 = \log .2179. \end{aligned}$$

5. $\log (.27031 \times .042809) = \log .27031 + \log .042809.$

$$\begin{aligned} \log .27031 &= 9.4319 - 10 \\ \log .042809 &= \underline{8.6315 - 10} \\ &\ 8.0634 - 10 = \log .01157. \end{aligned}$$

6. $\log (.063165 \times 11.134) = \log .063165 + \log 11.134.$

$$\begin{aligned} \log .063165 &= 8.8005 - 10 \\ \log 11.134 &= \underline{1.0466} \\ &\ 9.8471 - 10 = \log .7032. \end{aligned}$$

Result, $-.7032$.

7. $\log \dfrac{401.8}{52.37} = \log 401.8 - \log 52.37.$

$$\begin{aligned} \log 401.8 &= 2.6040 \\ \log 52.37 &= \underline{1.7191} \\ &\ 0.8849 = \log 7.672. \end{aligned}$$

8. $\log \dfrac{7.2321}{10.813} = \log 7.2321 - \log 10.813.$

$$\begin{aligned} \log\ 7.2321 &= 0.8592 \\ \log\ 10.813 &= \underline{1.0339} \\ & 9.8253 - 10 = \log .6688. \end{aligned}$$

9. $\log \dfrac{.3384}{.08659} = \log .3384 - \log .08659.$

$$\begin{aligned} \log\ .3384 &= 9.5294 - 10 \\ \log\ .08659 &= \underline{8.9374 - 10} \\ & 0.5920 \qquad = \log 3.908. \end{aligned}$$

Result, -3.908.

10. $\log \dfrac{9.163}{.0051422} = \log 9.163 - \log .0051422.$

$$\begin{aligned} \log\ 9.163 &= 0.9620 \\ \log .0051422 &= \underline{7.7112 - 10} \\ & 3.2508 \qquad = \log 1782. \end{aligned}$$

11. $\log \dfrac{22518}{64327} = \log 22518 - \log 64327.$

$$\begin{aligned} \log 22518 &= 4.3525 \\ \log 64327 &= \underline{4.8084} \\ & 9.5441 - 10 = \log .35. \end{aligned}$$

12. $\log \dfrac{.007514}{.015822} = \log .007514 - \log .015822.$

$$\begin{aligned} \log .007514 &= 7.8758 - 10 \\ \log .015822 &= \underline{8.1993 - 10} \\ & 9.6765 - 10 = \log .4748. \end{aligned}$$

Result, $-.4748$.

13. $\log \dfrac{3.3681}{12.853 \times .6349}$

$= \log 3.3681 + \operatorname{colog} 12.853 + \operatorname{colog} .6349.$

$$\begin{aligned} \log 3.3681 &= 0.5274 \\ \operatorname{colog} 12.853 &= 8.8910 - 10 \\ \operatorname{colog}\ .6349 &= \underline{0.1973} \\ & 9.6157 - 10 = \log .4127. \end{aligned}$$

14. $\log\dfrac{15.008\times .0843}{.06376\times 4.248}$

$= \log 15.008 + \log .0843 + \operatorname{colog} .06376 + \operatorname{colog} 4.248.$

$$\begin{aligned}
\log 15.008 &= 1.1763\\
\log .0843 &= 8.9258 - 10\\
\operatorname{colog} .06376 &= 1.1955\\
\operatorname{colog} 4.248 &= \underline{9.3718 - 10}\\
&\ 0.6694 \quad = \log 4.671.
\end{aligned}$$

Result, -4.671.

15. $\log\dfrac{2563\times .03442}{714.8\times .511}$

$= \log 2563 + \log .03442 + \operatorname{colog} 714.8 + \operatorname{colog} .511.$

$$\begin{aligned}
\log 2563 &= 3.4087\\
\log .03442 &= 8.5368 - 10\\
\operatorname{colog} 714.8 &= 7.1458 - 10\\
\operatorname{colog} .511 &= \underline{0.2916\qquad\ }\\
&\ 9.3829 - 10 = \log .2415.
\end{aligned}$$

16. $\log\dfrac{121.6\times 9.025}{48.3\times 3662\times .0856} = \log 121.6$

$+ \log 9.025 + \operatorname{colog} 48.3 + \operatorname{colog} 3662 + \operatorname{colog} .0856.$

$$\begin{aligned}
\log 121.6 &= 2.0850\\
\log 9.025 &= 0.9554\\
\operatorname{colog} 48.3 &= 8.3161 - 10\\
\operatorname{colog} 3662 &= 6.4363 - 10\\
\operatorname{colog} .0856 &= \underline{1.0675\qquad\ }\\
&\ 8.8603 - 10 = \log .0725.
\end{aligned}$$

Result, $-.0725$.

17. $\log (23.86)^3 = 3\times\log 23.86.$

$$\begin{aligned}
\log 23.86 &= 1.3777\\
&\ \underline{\qquad\quad 3}\\
&\ 4.1331 = \log 13587.
\end{aligned}$$

18. $\log (.532)^8 = 8\times\log .532.$

$$\begin{aligned}
\log .532 &= 9.7259 - 10\\
&\ \underline{\qquad\qquad\quad 8}\\
&\ 7.8072 - 10 = \log .006415.
\end{aligned}$$

19. $\log (1.0246)^7 = 7\times\log 1.0246.$

$$\begin{aligned}
\log 1.0246 &= 0.0105\\
&\ \underline{\qquad\quad 7}\\
&\ 0.0735 = \log 1.184.
\end{aligned}$$

Result, -1.184.

20. $\log(.09323)^5 = 5 \times \log .09323.$

$$\begin{array}{rl}\log\ .09323 & = 8.9695 - 10 \\ & \underline{5} \\ & 4.8475 - 10 = \log .000007038.\end{array}$$

21. $\log 5^{\frac{2}{3}} = \frac{2}{3}\log 5.$

$$\begin{array}{rl}\log 5 & = 0.6990;\ \times \frac{2}{3} = 0.4660 \\ & \qquad\qquad = \log 2.924.\end{array}$$

22. $\log(.8)^{\frac{2}{5}} = \frac{2}{5}\log .8.$

$$\begin{array}{rl}\log\ .8 & = 9.9031 - 10 \\ & \underline{2} \\ & 5)\underline{49.8062 - 50} \\ & \ \ 9.9612 - 10 = \log .9146.\end{array}$$

23. $\log(3.16)^{\frac{4}{3}} = \frac{4}{3}\log 3.16.$

$$\begin{array}{rl}\log\ 3.16 & = 0.4997;\ \times \frac{4}{3} = 0.6663 \\ & \qquad\qquad = \log 4.638.\end{array}$$

24. $\log(.021)^{\frac{5}{2}} = \frac{5}{2}\log .021.$

$$\begin{array}{rl}\log\ .021 & = 8.3222 - 10 \\ & \underline{5} \\ & 2)\underline{11.6110 - 20} \\ & \ \ 5.8055 - 10 = \log .0000639.\end{array}$$

25. $\log \sqrt{2} = \frac{1}{2}\log 2.$

$$\begin{array}{rl}\log\ 2 & = 0.3010;\ \div 2 = 0.1505 \\ & \qquad\qquad = \log 1.414.\end{array}$$

26. $\log \sqrt[4]{5} = \frac{1}{4}\log 5.$

$$\begin{array}{rl}\log\ 5 & = 0.6990;\ \div 4 = 0.1747 \\ & \qquad\qquad = \log 1.495.\end{array}$$

27. $\log \sqrt[5]{3} = \frac{1}{5} \log 3.$

$\log \quad 3 = 0.4771;\ \div 5 = 0.0954$
$= \log 1.246.$
Result, — 1.246.

28. $\log \sqrt{.4294} = \frac{1}{2} \log .4294.$

$\log \quad .4294 = 19.6329 - 20;\ \div 2 = 9.8164 - 10$
$= \log .6553.$

29. $\log \sqrt[3]{.02305} = \frac{1}{3} \log .02305.$

$\log \quad .02305 = 28.3626 - 30;\ \div 3 = 9.4542 - 10$
$= \log .2846.$

30. $\log \sqrt[8]{1000} = \frac{1}{8} \log 1000.$

$\log \quad 1000 = 3;\ \div 8 = 0.3750 = \log 2.372.$

31. $\log \sqrt[7]{.00951} = \frac{1}{7} \log .00951.$

$\log \quad .00951 = 67.9782 - 70;\ \div 7 = 9.7112 - 10$
$= \log .5142.$
Result, — .5142.

32. $\log \sqrt[5]{.0001011} = \frac{1}{5} \log .0001011.$

$\log \quad .0001011 = 46.0047 - 50;\ \div 5 = 9.2009 - 10$
$= \log .1588.$

35. $\log (2^{\frac{3}{2}} \times 3^{\frac{2}{3}}) = \frac{3}{2} \log 2 + \frac{2}{3} \log 3.$

$\log 2 = .3010;\ \times \frac{3}{2} = .4515$

$\log 3 = .4771;\ \times \frac{2}{3} = .3181$

$\overline{.7696} = \log 5.883.$

36. $\log \frac{3^{\frac{5}{8}}}{4^{\frac{2}{3}}} = \frac{5}{8} \log 3 - \frac{2}{3} \log 4.$

$\log 3 = .4771;\ \times \frac{5}{8} = \ .2982$

$\log 4 = .6021;\ \times \frac{2}{3} = \ .4014$

$\overline{9.8968 - 10} = \log .7885.$

37. $\log \dfrac{5^{\frac{3}{7}}}{10^{\frac{2}{9}}} = \dfrac{3}{7}\log 5 - \dfrac{2}{9}\log 10.$

$$\begin{aligned}
\log 5 &= .6990; \ \times\tfrac{3}{7} = .2996\\
\log 10 &= 1; \ \times\tfrac{2}{9} = \underline{.2222}\\
&\qquad\qquad\quad .0774 = \log 1.195.
\end{aligned}$$

38. $\log\left(\dfrac{6}{7}\right)^{\frac{5}{2}} = \dfrac{5}{2}(\log 6 - \log 7).$

$$\begin{aligned}
\log 6 &= .7782\\
\log 7 &= \underline{.8451}\\
&\ \ 9.9331 - 10\\
&\qquad\qquad\ \ \underline{5}\\
2)&\underline{19.6655 - 20}\\
&\ \ 9.8327 - 10 = \log .6803.
\end{aligned}$$

39. $\log\left(\dfrac{35}{113}\right)^{\frac{3}{8}} = \dfrac{3}{8}(\log 35 - \log 113).$

$$\begin{aligned}
\log 35 &= 1.5441\\
\log 113 &= \underline{2.0531}\\
&\ \ 9.4910 - 10\\
&\qquad\qquad\ \ \underline{3}\\
8)&\underline{78.4730 - 80}\\
&\ \ 9.8091 - 10 = \log .6443.
\end{aligned}$$

40. $\log\left(\dfrac{.08726}{.1321}\right)^{\frac{5}{3}} = \dfrac{5}{3}(\log .08726 - \log .1321).$

$$\begin{aligned}
\log .08726 &= 8.9408 - 10\\
\log .1321 &= \underline{9.1209 - 10}\\
&\ \ 9.8199 - 10\\
&\qquad\qquad\ \ \underline{5}\\
3)&\underline{29.0995 - 30}\\
&\ \ 9.6998 - 10 = \log .501.
\end{aligned}$$

41. $\log\sqrt[8]{\dfrac{21}{13}} = \dfrac{1}{8}(\log 21 - \log 13).$

$$\begin{aligned}
\log 21 &= 1.3222\\
\log 13 &= \underline{1.1139}\\
8)&\ \underline{.2083}\\
&\ \ .0260 = \log 1.062.
\end{aligned}$$

42. $\log\sqrt[9]{\frac{3}{7}}=\frac{1}{9}(\log 3-\log 7).$

$$\begin{array}{ll} \log 3 = & .4771 \\ \log 7 = & \underline{.8451} \\ & 9\overline{)89.6320-90} \\ & 9.9591-10=\log .9102. \end{array}$$

Result, $-.9102$.

43. $\log\left(\sqrt[5]{\frac{2}{3}}\div\sqrt[3]{\frac{3}{5}}\right)$

$$=\frac{1}{5}(\log 2-\log 3)-\frac{1}{3}(\log 3-\log 5).$$

$$\begin{array}{ll} \log 2 = & .3010 \\ \log 3 = & \underline{.4771} \\ & 5\overline{)49.8239-50} \\ & 9.9648-10 \\ & \underline{9.9260-10} \\ & .0388 \quad =\log 1.093. \end{array} \qquad \begin{array}{ll} \log 3 = & .4771 \\ \log 5 = & \underline{.6990} \\ & 3\overline{)29.7781-30} \\ & 9.9260-10 \end{array}$$

44. $\log(\sqrt[8]{2}\times\sqrt[5]{3}\times\sqrt[7]{.01})$

$$=\frac{1}{8}\log 2+\frac{1}{5}\log 3+\frac{1}{7}\log .01.$$

$$\begin{array}{lll} \log\ 2=.3010; & \div 8 = & .0376 \\ \log\ 3=.4771; & \div 5 = & .0954 \\ \log .01=68-70; & \div 7 = & \underline{9.7143-10} \\ & & 9.8473-10=\log .7035. \end{array}$$

45. $\log\sqrt[5]{\frac{3258}{49309}}=\frac{1}{5}(\log 3258-\log 49309).$

$$\begin{array}{ll} \log\ 3258 = & 3.5129 \\ \log 49309 = & \underline{4.6929} \\ & 5\overline{)48.8200-50} \\ & 9.7640-10=\log .5807. \end{array}$$

46. $\log\left(\frac{31.63}{429}\right)^{\frac{3}{17}}=\frac{3}{17}(\log 31.63-\log 429).$

$$\begin{array}{ll} \log 31.63 = & 1.5001 \\ \log\ 429 = & \underline{2.6325} \\ & 8.8676-10 \\ & \underline{\qquad\qquad 3} \\ & 17\overline{)166.6028-170} \\ & 9.8002-10 \quad =\log .6313. \end{array}$$

Result, $-.6313$.

47. $\log \dfrac{100^{\frac{2}{3}}}{(.7325)^{\frac{3}{7}}} = \dfrac{2}{3}\log 100 - \dfrac{3}{7}\log .7325.$

$$\begin{array}{ll}
\log\ 100 = 2; & \times\dfrac{2}{3} = 1.3333 \\
\log .7325 = 9.8648 - 10 & \\
\ 3 & \\
\hline
69.5944 - 70 \div 7 = & \underline{9.9421 - 10} \\
& 1.3912 = \log 24.62.
\end{array}$$

48. $\log \dfrac{\sqrt[3]{.0001289}}{\sqrt[4]{.0008276}} = \dfrac{1}{3}\log .0001289 - \dfrac{1}{4}\log .0008276.$

$$\begin{array}{ll}
\log .0001289 = 26.1103 - 30; & \div 3 = 8.7034 - 10 \\
\log .0008276 = 36.9178 - 40; & \div 4 = \underline{9.2294 - 10} \\
& \ 9.4740 - 10 \\
& = \log .2979.
\end{array}$$

49. $\log \dfrac{(.7469)^{\frac{5}{3}}}{(.2345)^{\frac{7}{2}}} = \dfrac{5}{3}\log .7469 - \dfrac{7}{2}\log .2345.$

$$\begin{array}{ll}
\log .7469 = 9.8732 - 10 & \log .2345 = 9.3701 - 10 \\
\underline{5} & \underline{7} \\
3\underline{)29.3660 - 30} & 2\underline{)15.5907 - 20} \\
9.7887 - 10 & 7.7953 - 10 \\
\underline{7.7953 - 10} & \\
1.9934 = \log 98.5. &
\end{array}$$

50. $\log \dfrac{\sqrt[11]{.0073}}{(.68291)^{\frac{5}{2}}} = \dfrac{1}{11}\log .0073 - \dfrac{5}{2}\log .68291.$

$$\begin{array}{lll}
& \log .0073 = 107.8633 - 110 & \log .68291 = 9.8343 - 10 \\
\text{Dividing by } 11, = & 9.8058 - 10 & \underline{5} \\
& \underline{9.5857 - 10} & 2\underline{)19.1715 - 20} \\
& .2201 & 9.5857 - 10 \\
& = \log 1.66. &
\end{array}$$

51. $\log \dfrac{\sqrt{5.955} \times \sqrt[3]{61.2}}{\sqrt[5]{298.54}}$

$$= \frac{1}{2}\log 5.955 + \frac{1}{3}\log 61.2 + \frac{1}{5}\operatorname{colog} 298.54.$$

$$\begin{array}{ll}
\log\ 5.955 = 0.7748 & ; \div 2 = 0.3874 \\
\log\ 61.2 = 1.7868 & ; \div 3 = 0.5956 \\
\operatorname{colog} 298.54 = 47.5250 - 50; & \div 5 = \underline{9.5050 - 10} \\
& \ 0.4880 \quad = \log 3.076.
\end{array}$$

52. $\log(538.2 \times .0005969)^{\frac{1}{8}} = \frac{1}{8}(\log 538.2 + \log .0005969).$

$$\begin{array}{rl}
\log \quad 538.2 = & 2.7310 \\
\log .0005969 = & \underline{6.7759 - 10} \\
8)\overline{79.5069 - 80} & \\
& 9.9384 - 10 = \log .8678.
\end{array}$$

53. $\log[(18.9503)^{11} \times (.1)^{14}] = 11 \times \log 18.9503 + 14 \times \log .1.$

$$\begin{array}{l}
\log 18.9503 = 1.2777;\ \times 11 = 14.0547 \\
\log \quad .1 = 9 - 10;\ \times 14 = \underline{6. \qquad - 20} \\
\qquad\qquad\qquad\qquad\qquad .0547 \qquad = \log 1.134.
\end{array}$$

54. $\log \sqrt[6]{3734.9 \times .00001108} = \frac{1}{6}(\log 3734.9 + \log .00001108).$

$$\begin{array}{rl}
\log \quad 3734.9 = & 3.5723 \\
\log .00001108 = & \underline{5.0445 - 10} \\
6)\overline{58.6168 - 60} & \\
& 9.7695 - 10 = \log .5881.
\end{array}$$

55. $\log[(2.6317)^{\frac{3}{4}} \times (.71272)^{\frac{2}{5}}]$

$$= \frac{3}{4}\log 2.6317 + \frac{2}{5}\log .71272.$$

$$\begin{array}{rlcrl}
\log 2.6317 = & .4203 & \qquad & \log .71272 = & 9.8529 - 10 \\
& \underline{3} & & & \underline{2} \\
4)\overline{1.2609} & & & 5)\overline{49.7058 - 50} & \\
& .3152 & & & 9.9412 - 10 \\
& \underline{9.9412 - 10} & & & \\
& .2564 \qquad = \log 1.805. & & &
\end{array}$$

56. $\log \dfrac{\sqrt[3]{.008193} \times (.06285)^{\frac{3}{2}}}{.98342}$

$$= \frac{1}{3}\log .008193 + \frac{3}{2}\log .06285 + \text{colog } .98342.$$

$$\begin{array}{l}
\log .008193 = 27.9134 - 30;\ \div 3 = 9.3045 - 10 \\
\log \ .06285 = \ 8.7983 - 10 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{3} \\
\qquad\qquad\quad 16.3949 - 20;\ \div 2 = 8.1974 - 10 \\
\qquad\qquad\qquad \text{colog } .98342 = \underline{0.0072 \qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 7.5091 - 10 \\
\qquad\qquad\qquad\qquad\qquad\quad = \log .003229.
\end{array}$$

57. $\log(\sqrt{.035} \times \sqrt[6]{.62667} \times \sqrt[3]{.0072103})$

$$= \frac{1}{2}\log .035 + \frac{1}{6}\log .62667 + \frac{1}{3}\log .0072103.$$

$$\begin{aligned}
\log\ .035 &= 18.5441 - 20;\ \div 2 = 9.2720 - 10\\
\log\ .62667 &= 59.7971 - 60;\ \div 6 = 9.9662 - 10\\
\log .0072103 &= 27.8579 - 30;\ \div 3 = \underline{9.2860 - 10}\\
&\qquad\qquad\qquad\qquad\quad 8.5242 - 10\\
&\qquad\qquad\qquad\qquad = \log .03344.
\end{aligned}$$

Art. 362. — Page 305.

3.
$$11^x = 3.$$
$$x \log 11 = \log 3.$$
$$\therefore x = \frac{\log 3}{\log 11} = \frac{.4771}{1.0414} = .4581.$$

4.
$$.3^x = .8.$$
$$x \log .3 = \log .8.$$
$$\therefore x = \frac{\log .8}{\log .3} = \frac{9.9031 - 10}{9.4771 - 10}$$
$$= \frac{-.0969}{-.5229} = .1853.$$

5.
$$13^x = .281.$$
$$x \log 13 = \log .281.$$
$$\therefore x = \frac{\log .281}{\log 13} = \frac{9.4487 - 10}{1.1139}$$
$$= \frac{-.5513}{1.1139} = -.4949.$$

6.
$$.703^x = 1.096.$$
$$x \log .703 = \log 1.096.$$
$$\therefore x = \frac{\log 1.096}{\log .703} = \frac{.0398}{9.8470 - 10}$$
$$= \frac{.0398}{-.153} = -.2601.$$

7.
$$a^x = b^m c^n.$$
$$x \log a = m \log b + n \log c.$$
$$\therefore x = \frac{m \log b + n \log c}{\log a}.$$

8.
$$ma^{\frac{1}{x}} = n.$$
$$\log m + \frac{1}{x}\log a = \log n.$$
$$\frac{1}{x}\log a = \log n - \log m.$$
$$\therefore x = \frac{\log a}{\log n - \log m}.$$

9. By Art. 319, $l = ar^{n-1}$.
$$\therefore \log l = \log a + (n-1)\log r.$$
$$\log l - \log a = (n-1)\log r.$$
$$n - 1 = \frac{\log l - \log a}{\log r}.$$
$$\therefore n = \frac{\log l - \log a}{\log r} + 1.$$

10. By Art. 320, $S = \dfrac{ar^n - a}{r-1}$.
$$(r-1)S = ar^n - a.$$
$$\therefore r^n = \frac{(r-1)S + a}{a}.$$
$$\therefore n\log r = \log[(r-1)S + a] - \log a.$$
$$\therefore n = \frac{\log[(r-1)S + a] - \log a}{\log r}.$$

11. By Ex. 13, p. 277, $r = \dfrac{S-a}{S-l}$.
$$\therefore \log r = \log(S-a) - \log(S-l).$$
Substituting in the result of Ex. 9, p. 305,
$$\therefore n = \frac{\log l - \log a}{\log(S-a) - \log(S-l)} + 1.$$

12. By Ex. 14, p. 277, $a = rl - (r-1)S$.
Substituting in the result of Ex. 9, p. 305,
$$\therefore n = \frac{\log l - \log[rl - (r-1)S]}{\log r} + 1.$$